Spectral Theory and the Gelfand Transform

by

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Abstract

The overall goal of this thesis is to study spectral and Gelfand theory as it relates to unital Banach and C^*-algebras. In the first part, we develop the necessary algebraic, analytic, and topological background relevant to the content of this work. We also discuss concrete examples of algebras frequently used in the subsequent sections.

In the second part of this thesis, we develop spectral theory by first defining the spectrum of an algebra through the characterization of the invertible and non-invertible elements. In particular, we establish properties of the commutative unital Banach algebra $\ell^1(\mathbb{Z})$. We also establish fundamental results such as Gelfand's spectral radius formula and prove the Gelfand-Mazur theorem.

In the last part of this thesis, we study the spectrum, also known as the maximal ideal space, of commutative unital Banach algebras. Through this, we develop the Gelfand transform, which maps elements of the algebra to continuous functions on its spectrum.

Chapter 1

Introduction

The foundations of harmonic analysis rely heavily on spectral theory and the Gelfand transform, as they provide an important connection between abstract algebraic structures and concrete function spaces. This thesis specifically explores this connection from unital Banach and C^* algebras and their analytic representation via the Gelfand transform.

Informally, the Gelfand transform maps elements of a commutative Banach algebra $\mathcal{A}$ to the space of continuous functions on its spectrum. One specific instance of the Gelfand transform is the Fourier transform, as we will see in the last chapter . We note that the Gelfand transform is applicable to commutative Banach algebras, while the Fourier transform is applicable to locally compact Abelian groups.

In the first part of this thesis, Unital Banach Algebras and Spectral Theory, we establish results relating to the spectrum of an algebra. We begin by examining invertibility in Banach algebras, proving results such as the Neumann series expansion for invertible elements. Other notable results include Gelfand's spectral radius formula and the Gelfand-Mazur theorem, which establishes that a Banach algebra where every non-zero element is invertible is isomorphic to $\mathbb{C}$.

In the second part of this thesis, Gelfand Theory, we develop the representation theory of commutative Banach algebras. We begin by exploring properties and the behavior of multiplicative functionals and build up to a one-to-one correspondence between the set of maximal ideals of an algebra and the algebra's spectrum. We then leverage this correspondence to prove key results such as the Gelfand-Naimark theorem. We conclude this chapter with a brief commentary of how much of the work in this thesis can be extended from unital algebras to non-unital algebras.

Lastly, unless stated otherwise, the content presented here is an elaboration of material drawn from [4], A Course in Abstract Harmonic Analysis, by Folland.

Chapter 2

Background and Notation

In this section, we present key background information essential for understanding the Gelfand transform, a generalization of the Fourier transform. In the first section, we review the algebraic concepts and propositions necessary for the development of this thesis. Similarly, the second section discusses the necessary background for functional analysis and the third for topology. In each section, we begin with a few definitions and build up to results that will be used in the subsequent chapters.

2.1 Algebra

We begin by defining a partial ordering, which provides a way to compare elements within a set.

Definition 2.1. A *partial ordering* on a set X, denoted $\leq$, is a binary relation on X, that is

1. Reflexive: $x \leq x$ for all $x \in X$;
2. Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$;
3. Transitive: If $a \leq b$ and $b \leq c$ then $a \leq c$.

When a set X is equipped with a partial ordering, we say X is a partially ordered set. The term *partially* ordered implies that X may contain elements x and y for which neither $x \leq y$ nor $y \leq x$ holds. In such cases, we say x and y are incomparable.

Otherwise, if $x \leq y$ or $y \leq x$ hold, we say x and y are comparable elements. This leads to the following definition.

Definition 2.2. A subset S of X is called a *chain* (totally ordered set) if any two elements $x, y \in S$ are comparable. I.e., either $x \leq y$ or $y \leq x$.

That is, a chain is a partially ordered set that contains no incomparable elements. These play a central role in many areas of mathematics, including set theory and algebra.

Moreover, for a subset S of a partially ordered set X, we say an element $u \in X$ is an upper bound of S if for all $x \in S$ we have $x \leq u$. Depending on our sets S and X, such a u may not exist. Furthermore, we say an element is $m \in X$ is a maximal element of X if $m \leq x$ implies $m = x$ for some $x \in X$. Similarly, X may not contain a maximal element.

For example, consider the space $\mathbb{R}$ where $\leq$ has its usual meaning. Here, X is totally ordered but has no maximal element. In contrast, consider the power set of a given set X, denoted $\mathscr{P}(X)$, equipped with the partial ordering $\leq$ defined by inclusion. That is, $A \leq B$ is and only if $A \subseteq B$. Then, $\mathscr{P}(X)$ is partially ordered and the only maximal element is X.

Using these concepts, we can now state Zorn's lemma. This is a powerful result provides a way to establish the existence of maximal elements.

Lemma 2.3 (Zorn's lemma [7], 41-6.). *If every chain in a partially ordered set $\mathcal{S}$ has an upper bound, then $\mathcal{S}$ has at least one maximal element.*

With these foundational concepts in place, we now turn to algebraic structures, beginning with rings and ring homomorphisms. These will be essential for understanding the relationship between the spectrum of an algebra and its maximal ideals, as explored in Chapter 4.

Definition 2.4. Let $\mathcal{R}, \mathcal{S}$ be rings. A *ring homomorphism* is a map $\varphi : \mathcal{R} \to \mathcal{S}$ that respects addition and multiplication of both the domain and codomain of φ. That is, for all $a, b \in \mathcal{R}$,

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi(b).$$

Consequently, ring homomorphisms also preserve multiplicative identities, i.e.,

$$\varphi(1_{\mathcal{R}}) = 1_{\mathcal{S}}.$$

We define the *kernel* of a homomorphism to be the set $\{x \in \mathcal{R} : \varphi(x) = 1_{\mathcal{S}}\}$. This is the set of all elements in our ring that map to the identity element. Moreover, ring homomorphisms are closely tied to ideals, since every ideal is the kernel of a ring homomorphism.

Theorem 2.5 ([6], 15.4.). *Every ideal of a ring $\mathcal{R}$ is the kernel of a ring homomorphism of $\mathcal{R}$. In particular, an ideal $\mathcal{A}$ is the kernel of the mapping $r \to r + \mathcal{A}$ from $\mathcal{R}$ to $\mathcal{R}/\mathcal{A}$.*

The connection between ring homomorphisms and ideals is further developed by the First Isomorphism theorem. This establishes a natural isomorphism between the quotient of a ring by the kernel of a homomorphism and the image of that homomorphism.

Theorem 2.6 (First Isomorphism theorem [6], 15.3.). *Let ϕ be a ring homomorphism from $\mathcal{R}$ to $\mathcal{S}$. Then the mapping from $\mathcal{R}/\ker(\phi)$ to $\phi(\mathcal{R})$, given by $r + \ker(\phi) \to \phi(r)$ is an isomorphism.*

These theorems demonstrate an important connection between ring homomorphisms and their ideals, which again will be of great importance in Chapter 4. We further explore this relationship with the following.

Theorem 2.7 (Lattice Isomorphism theorem [3], Theorem 20.). *Let I be an ideal of a ring R.*

1. *If $\mathcal{J}$ is an ideal of $\mathcal{R}$ that contains $\mathcal{I}$, then $\mathcal{J}/\mathcal{I}$ is an ideal of $\mathcal{R}/\mathcal{I}$.*
2. *If $\tilde{J}$ is an ideal of $\mathcal{R}/\mathcal{I}$, then $\tilde{\mathcal{J}} = \mathcal{J}/\mathcal{I}$ for some ideal J of R containing I.*

The next results, which characterize maximal ideals in commutative rings, will be key tools in establishing a one-to-one correspondence between the spectrum of an algebra and the set of maximal ideals.

Proposition 2.8. *Let $\mathcal{R}$ be a commutative ring with identity. An ideal $\mathcal{M}$ is maximal if and only if $\mathcal{R}/\mathcal{M}$ is a field.*

Proof. First, let $\mathcal{M}$ be a maximal ideal of a commutative ring $\mathcal{R}$. Then the only ideals of $\mathcal{R}$ that contain $\mathcal{M}$ are $\mathcal{M}$ and $\mathcal{R}$ itself. Thus, by the Lattice Isomorphism theorem, the only ideals of $\mathcal{R}/\mathcal{M}$ are $\mathcal{R}/\mathcal{M}$ and $\mathcal{M}/\mathcal{M} = \{0+\mathcal{M}\}$. Therefore, $\mathcal{R}/\mathcal{M}$ is a field.

Conversely, suppose $\mathcal{R}/\mathcal{M}$ is a field. Then the only ideals of $\mathcal{R}/\mathcal{M}$ are $\mathcal{R}/\mathcal{M}$ and $\{0+\mathcal{M}\}$. Again, by the Lattice Isomorphism theorem, the only ideals of $\mathcal{R}$ containing $\mathcal{M}$ are $\mathcal{R}$ and $\mathcal{M}$. Thus, $\mathcal{M}$ is a maximal ideal of $\mathcal{R}$. ∎

Proposition 2.9. *Any commutative ring $\mathcal{R}$ has a maximal ideal.*

Proof. Let $\mathscr{F}$ be the set of all proper ideals of $\mathcal{R}$, and define a partial ordering of $\mathscr{F}$ given by inclusion. That is, for $F_1, F_2 \in \mathscr{F}$, we have $F_1 \leq F_2$ whenever $F_1 \subseteq F_2$. Also note that $\mathscr{F}$ is nonempty as it contains the zero ideal. Suppose that $C = \{I_\alpha\}_{\alpha\in A}$ is a chain in $\mathscr{F}$. Then $\mathcal{J} = \bigcup_{\alpha\in A} I_\alpha$ is an ideal of $\mathcal{R}$ and, toward a contradiction, suppose that $\mathcal{J}$ is not a proper ideal of $\mathcal{R}$. Then $1 \in \mathcal{J}$, which implies that $1 \in I_{\alpha_0}$ for some $\alpha_0 \in A$. Moreover, this implies that $I_{\alpha_0} = \mathcal{R}$, contradicting that I_α is proper.

Thus, if $C = \{I_\alpha\}_{\alpha\in A}$ is a chain in $\mathscr{F}$, then $\mathcal{J} = \bigcup_{\alpha\in A} I_\alpha$ is an upper bound in $\mathscr{F}$. By Lemma 2.3 (Zorn's lemma), there exists a maximal element in $\mathscr{F}$ which is a maximal ideal of $\mathcal{R}$. ∎

Next, we define an algebraic structure that will be of great importance in this thesis, an algebra. Algebras provide a framework for studying operations on elements that combine algebraic and analytic properties.

DEFINITION 2.10. An *algebra* A over a field $\mathbb{F}$ is a collection $(A, +, *, \mathbb{F})$, satisfying

1. $(A, +, \mathbb{F})$ is a vector space,
2. multiplication $*$ is bilinear,
3. for all $x, y \in A$ and $\alpha \in \mathbb{F}$, $(\alpha x) * y = x * (\alpha y) = \alpha(x * y)$.

An example of an algebra, which we will see in Chapters 3 and 4, is $\ell^1(\mathbb{Z})$. This is the space of bounded complex-valued sequences $x = (x_n)_{i=1}^{\infty}$ for which $\sum_{i=1}^{\infty} |x_i| < \infty$.

EXAMPLE 2.11. ($\mathcal{A} = \ell^1(\mathbb{Z})$ is an algebra).

Suppose $\mathcal{A} = \ell^1(\mathbb{Z})$ with multiplication defined as convolution. That is, for $a, b \in \ell^1(\mathbb{Z})$ the product of a and b is defined as $a * b = c$, where $c_n = \sum_{m=-\infty}^{\infty} a_m b_{n-m}$. To show convolution is closed, let $x, y \in \ell^1(\mathbb{Z})$. Then,

$$\begin{aligned}
\|x * y\|_1 = \sum_{n=-\infty}^{\infty} |(x * y)_n| = \sum_{n=-\infty}^{\infty} \Big| \sum_{m=-\infty}^{\infty} x_m y_{n-m} \Big| &\leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |x_m y_{n-m}| \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |x_m||y_{n-m}| \\
&= \sum_{m=-\infty}^{\infty} |x_m| \sum_{n=-\infty}^{\infty} |y_{n-m}| \\
&= \sum_{m=-\infty}^{\infty} |x_m| \sum_{\substack{k=-\infty \\ k=n-m}}^{\infty} |y_k| \\
&= \|x\|_1 \|y\|_1,
\end{aligned}$$

where the fourth equality will follow by Theorem 2.15 (Fubini's theorem for infinite series), since the series are absolutely convergent. Thus, convolution is closed.

Trivially, $\ell^1(\mathbb{Z})$ is a vector space. To show multiplication is bilinear, let $\alpha, \beta \in \mathbb{C}$ and $f, g, h \in \mathcal{A}$. Then,

$$\begin{aligned}((\alpha f + \beta g) * h)_n = \sum_{m=-\infty}^{\infty} (\alpha f + \beta g)_m h_{n-m} &= \sum_{m=-\infty}^{\infty} \alpha f_m h_{n-m} + \beta g_m h_{n-m} \\ &= \alpha \sum_{m=-\infty}^{\infty} f_m h_{n-m} + \beta \sum_{m=-\infty}^{\infty} g_m h_{n-m} \\ &= \alpha(f * h)_n + \beta(g * h)_n.\end{aligned}$$

Similarly, $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$. Thus multiplication, when defined as convolution, is bilinear.

Next, we show that scalar multiplication respects convolution. Let $x, y \in \ell^1(\mathbb{Z})$ and $\alpha \in \mathbb{C}$. Then,

$$((\alpha x) * y)_n = \sum_{k=-\infty}^{\infty} \alpha x_k y_{n-k} = \alpha \sum_{k=-\infty}^{\infty} x_k y_{n-k} = \alpha(x * y)_n$$

and

$$(x * (\alpha y))_n = \sum_{k=-\infty}^{\infty} x_k (\alpha y_{n-k}) = \alpha \sum_{k=-\infty}^{\infty} x_k y_{n-k} = \alpha(x * y)_n.$$

Therefore, $\mathcal{A} = \ell^1(\mathbb{Z})$ is an algebra.

2.2 Functional Analysis

The study of normed spaces and Banach spaces is central to functional analysis. These spaces provide a framework for analyzing functions and operators in infinite-dimensional settings.

Definition 2.12. A *normed space* X is a vector space equipped with a norm. Further, a *Banach space* is a complete normed space, with respect to the metric defined by the norm.

One of the key results in functional analysis is the Uniform Boundedness Principle, which provides conditions under which a family of bounded linear operators is uniformly bounded.

THEOREM 2.13 (Uniform Boundedness Principle [5], 5.12.). *Suppose X and Y are normed vector spaces and $\mathscr{A}$ is a subset of $\mathscr{L}(X,Y)$, the space of all bounded linear maps from X to Y. If X is a Banach space and $\sup\limits_{T\in\mathscr{A}} ||Tx|| < \infty$, for all $x \in X$, then $\sup\limits_{T\in\mathscr{A}} ||T|| < \infty$.*

This result is particularly useful in Chapter 3, as it aids us in establishing Gelfand's spectral radius formula. We will also use the next theorem to construct examples of unital Banach algebras.

THEOREM 2.14 (Fubini's theorem [1], 5.32.). *Let (X,μ) and (Y,ν) be measure spaces, and let $f : X \times Y \to \mathbb{C}$ be a measurable function. If f is integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$, then,*

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y).$$

That is, the iterated integral and the double integrals agree.

Analogously, we have a similar result for series instead of integrals.

THEOREM 2.15 ([1], 5.31.). *Let $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ such that $\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} f(n,m)$ is absolutely convergent. Then we have,*

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n,m) = \sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} f(n,m) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} f(n,m) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(n,m).$$

Our last result contains some basic facts about the convolution of L^p functions, where $f * g$ denotes the convolution of $f, g \in L^p$.

PROPOSITION 2.16 (Young's inequality [5], 8.7.). *If $f \in L^1$ and $g \in L^p$ with $1 \le p \le \infty$, then $f * g(x)$ exists for almost every x, $f * g \in L^p$, and $\|f * g\|_p \le \|f\|_1 \cdot \|g\|_p$.*

2.3 Topology

Topological spaces provide a general framework for studying continuity, convergence, and compactness. The following definitions and results will be essential for our later discussions.

A *topology* on a set X is a collection τ of subsets of X, satisfying

1. the sets $\emptyset$ and X belong to τ,

2. any union of elements of τ belongs to τ, and

3. any finite intersection of elements of τ belongs to τ.

A *topological space* is a set equipped with a topology, denoted (X, τ). Further, the elements of a topology τ are what we consider to be open sets in X.

DEFINITION 2.17. If X is a topological space, then a *base* B for X is a collection of open sets in X such that every open set is a union of open sets from B.

DEFINITION 2.18. A collection of sets is a *subbase* for a topology on X if it forms a base after all possible finite intersections are added.

By defining a topology in terms of a base or subbase, we can efficiently describe its structure while minimizing redundancy. Given a subbase $\mathcal{S}$ for a topology on a set X, the topology τ generated by $\mathcal{S}$ is constructed in two steps:

1. We form a base $\mathcal{B}$ by taking all finite intersections of elements of $\mathcal{S}$.

2. Then we generate the topology τ by taking arbitrary unions of elements of $\mathcal{B}$.

This ensures that τ is the coarsest topology containing $\mathcal{S}$. Some topologies that are of particular interest are the weak and weak* topologies.

DEFINITION 2.19. Let X be a set and X_α a topological space with $f_\alpha : X \to X_\alpha$, for each $\alpha \in A$. The *weak topology* on X, induced by the collection $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ of functions, is the weakest topology on X making each f_α continuous. We denote this topology by $\sigma(X, \mathcal{F})$.

This is the topology on X for which the sets $f_\alpha^{-1}(U_\alpha)$, where $\alpha \in A$ and U_α is open in X_α, form a subbase. That is, $\sigma = \{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \tau_\alpha, \alpha \in A\}$. Note, for a normed-vector space X, the weak topology on X is $\sigma(X, X^*)$. Here, X^* denotes the dual space of X, the space of all continuous linear functionals on X. Similarly, the weak* topology is $\sigma(X^*, X)$, where we consider $X \subset X^{**}$, the double dual of X.

Next, we discuss the notion of convergence in a topological space. For this, we need the following.

DEFINITION 2.20. A *net* in a set X is a function $P : \Lambda \to X$, where Λ is some directed set. We denote the point $P(\lambda)$ by x_λ, and we will denote the net by $(x_\lambda)_{\lambda \in \Lambda}$.

We say a set Λ is a *directed set* if there is a relation $\leq$ on Λ that satisfies

1. $\lambda \leq \lambda$ for each $\lambda \in \Lambda$,
2. if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$,
3. if $\lambda_1, \lambda_2 \in \Lambda$ then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

This relation is referred to as a *direction* on Λ, or is said to direct Λ. Using this, we can now discuss convergence in a topological space.

DEFINITION 2.21. Let (x_λ) be a net in a space X. Then (x_λ) *converges* to $x \in X$, denoted $x_\lambda \to x$, if for each neighborhood U of x, there is some $\lambda_0 \in \Lambda$ such that whenever $\lambda \geq \lambda_0$ we have $x_\lambda \in U$.

Thus, $x_\lambda \to x$ if and only if each neighborhood of x contains a tail of (x_λ). We say (x_λ) converges weakly to $x \in X$ if for every $f \in X^*$ we have $f(x_\lambda) \to f(x)$. Similarly, $f_n \to f$ in the weak* topology if and only if $f_n(x) \to f(x)$ for all $x \in X$. Convergence of nets will be a useful tool later in Chapter 4. Our next theorem will provide us important information about the spectrum of an algebra.

THEOREM 2.22 (Banach-Alaoglu theorem [5], 5.18.). *If X is a normed space, then the closed unit ball in the continuous dual space X^* is compact with respect to the weak* topology.*

We now turn to an important tool, which allows us to establish equivalences between topological spaces.

DEFINITION 2.23. Let X and Y be topological spaces. A *homeomorphism* $T : X \to Y$ is a continuous, bijective map such that $T^{-1} : Y \to X$ is also continuous.

By establishing a homeomorphism between a compact Hausdorff space and the spectrum of the algebra of continuous functions on that space, we can bridge the gap between topological and algebraic structures.

THEOREM 2.24 ([8], 7.9.). *If X and Y are topological spaces and $f : X \to Y$ is one-to-one and onto, then the following are equivalent:*

1. *the function f is a homeomorphism,*

2. *if $G \subset X$, then $f(G)$ is open in Y if and only if G is open in X,*

3. *if $F \subset X$, then $f(F)$ is closed in Y if and only if F is closed in X.*

Proof. First, we will show that 1 implies 2. Suppose that f is a homeomorphism and $f(G) \subset Y$ is open. As f is continuous, and the pre-image of an open set is open, it follows that $G \subset X$ is open. Conversely, suppose that $G \subset X$ is open. As f^{-1} is continuous, it follows that the pre-image of G under f^{-1}, namely $f(G)$, is open in Y. Thus, 1 implies 2.

Next, we will show that 2 implies 3. Assume for any $G \subset X$, we have $f(G) \subset Y$ is open if and only if $G \subset X$ is open. Suppose that $F \subset X$ is closed. Then $X \setminus F$ is open in X, so it follows that $f(X \setminus F) = Y \setminus f(F)$ is open in Y. Thus, $f(F)$ is closed in Y. Conversely, if $f(F) \subset Y$ is closed, then $Y \setminus f(F)$ is open and so $f^{-1}(Y \setminus f(F)) = X \setminus F$ is open in X. Therefore, F is closed in X, and so we conclude 2 implies 3.

Lastly, we will show 3 implies 1. Assume for any $F \subset X$, we have $f(F)$ is closed in Y if and only if F is closed in X. Then, for any closed $F \subset Y$, as f is bijective, $f^{-1}(F) \subset X$. By assumption, $f(f^{-1}(F)) = F$ is closed in Y if and only if $f^{-1}(F)$ is closed in X. Thus, the preimage of a closed set in Y under f, is closed in X, so f is continuous.

Similarly, if $F \subset X$ is closed, then $f(F) \subset Y$ is closed. Since f is bijective, we have $(f^{-1})^{-1}(F) = f(F)$. Thus, $(f^{-1})^{-1}(F)$ is closed in Y whenever F is closed in X, and so f^{-1} is continuous. Therefore, f is a homeomorphism, and so 3 implies 1. ■

The last proposition in this section, which characterizes topological spaces, will help us establish a homeomorphism between a compact Hausdorff space and the spectrum on the set of continuous functions of that space. To proceed, we first define a normal topological space.

Definition 2.25. A topological space X is *normal* if whenever A and B are disjoint closed sets in X, there are disjoint open sets U and V with $A \subset U$ and $B \subset V$.

In a normal space, if two closed sets are disjoint (that is, they do not overlap), there always exist two disjoint open sets that separate them. This property represents a stronger form of separation than what is provided in, for example, a Hausdorff space (where any two distinct points can be separated by disjoint open sets) or a regular space (where a closed set and a point not in the set can be separated by disjoint open sets). Before our last result, we also need the following definitions.

Definition 2.26. If (X, τ) is a topological space and $A \subset X$, the collection

$$\tau' = \{G \cap A : G \in \tau\}$$

is a topology for A called the *subspace topology* or the relative topology.

Definition 2.27. Let X and Y be topological spaces, and let $f : X \to Y$. Then

f is *continuous* at $x_0 \in X$ if for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $f(U) \subset V$. We say f is *continuous* on X if f is continuous at each $x_0 \in X$.

With these definitions, we are able to establish a powerful tool that we will later use to establish a homeomorphism between a compact Hausdorff space X and the spectrum of the set of continuous functions on X.

PROPOSITION 2.28 (Urysohn's lemma [8]), 15.6.). *A topological space X is normal if and only if whenever A and B are disjoint closed sets in X, there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Suppose X is normal, $A, B \subset X$ are closed, disjoint sets in X, and $U_1 = X \setminus B$. Since X is normal, let $U_{1/2}$ be an open set such that

$$A \subset U_{1/2} \quad \text{and} \quad \overline{U}_{1/2} \subset U_1.$$

Note that $\overline{U}_{1/2} \cap B = \emptyset$. This implies that A and $X \setminus U_{1/2}$ are disjoint, closed sets and so are $\overline{U}_{1/2}$ and B. Thus, there exist open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4}, \quad \overline{U}_{1/4} \subset U_{1/2}, \quad \overline{U}_{1/2} \subset U_{3/4}, \quad \text{and} \quad \overline{U}_{3/4} \subset U_1$$

and so $\overline{U}_{3/4} \cap B = \emptyset$.

Now, suppose sets $U_{k/2^n}$, for $k = 1, \ldots, 2^n - 1$ are defined such that

$$A \subset U_{1/2^n}, \ldots, \overline{U}_{k-1/2^n} \subset U_{k/2^n}, \ldots, \overline{U}_{(2^n-1)/2^n} \subset U_1$$

and so $\overline{U}_{(2^n-1)/2^n} \cap B = \emptyset$. By normality, the process can be continued to provide sets $U_{k/2^{n+1}}$, $k = 1, \ldots, 2^{n+1} - 1$ defined similarly. Then for each dyadic rational r we have an open set U_r such that

1. $A \subset U_r$ and $\overline{U}_r \cap B = \emptyset$, and

2. $\overline{U}_r \subset U_s$ whenever $r < s$.

Define $f : X \to [0,1]$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ belongs to no } U_r \\ \inf\{r : x \in U_r\} & \text{otherwise .} \end{cases}$$

Clearly, $f(A) = 0$ and $f(B) = 1$. Next, we will prove two claims that we will use to show f is continuous.

First, if $x \in \overline{U}_p$ then $f(x) \leq p$. To show this, let $x \in \overline{U}_p$. Then $x \in \overline{U}_p \subset U_q$ for all dyadic rationals $q > p$ and so $f(x) = \inf\{r : x \in U_r\}$. Thus, $\inf\{r : x \in U_r\} \leq p$.

Next, if $x \notin U_p$, then $f(x) \geq p$. Suppose $x \notin U_q$ for any $q \leq p$. If there exists no dyadic rational r such that $x \in U_r$, then we have $f(x) = 1$. If there exists at least one dyadic rational $r > p$ such that $x \in U_r$, it follows that $f(x) = \inf\{r : x \in U_r\} \geq p$. In either case, we see that $f(x) \geq p$.

Now, we will show that f is continuous. To that end, suppose that $U = (a, b)$ is an open interval in $\mathbb{R}$ which intersects $[0,1]$. We note that this implies $[0,1] \cap (a,b)$ is an open set in the subspace topology on $[0,1]$. We wish to show $f^{-1}(U)$ is open in X. Thus, fix $x \in f^{-1}(U)$ and let V be an open set in X such that $x \in V \subset f^{-1}(U)$. Thus, $f(x) \in f(V) \subset U = (a,b)$.

So, for the fixed $x \in f^{-1}(U)$, we have $f(x) \in U = (a,b)$. Thus, we can find rational p and q such that

$$a < p < f(x) < q < b.$$

Since $p < f(x)$, it follows that $x \notin \overline{U}_p$ by the contrapositive of our first claim. On the other hand, as $f(x) < q$, by the contrapositive of our second claim we have $x \in U_q$. Thus, as $x \notin \overline{U}_p$ and $x \in U_q$, it follows that $x \in U_q \setminus \overline{U}_p$. Take $V = U_q \setminus \overline{U}_p$ to be our open set.

Lastly, we will show $V \subset (a, b)$. Then, for a fixed $y \in V$ we have $y \in U_q \subset \overline{U}_q$. So by our first claim, we have $f(y) \leq q < b$. Since $y \notin \overline{U}_p \supset U_p$ we also have $f(y) \geq p > a$ by the second claim. Therefore, $f(y) \in [p, q] \subset (a, b)$ and so f is continuous.

Conversely, suppose A and B are disjoint closed sets in X and let $f : X \to [0, 1]$ be a continuous function such that $f(A) = 0$ and $f(B) = 1$. Then,

$$A \subseteq f^{-1}\left([0, \frac{1}{2})\right) \quad \text{and} \quad B \subseteq f^{-1}\left((\frac{1}{2}, 1]\right).$$

Note that the preimages $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint as f is continuous and they are open as $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are open in the subspace topology. Therefore, we conclude that X is normal. ■

Chapter 3

Unital Banach Algebras and Spectral Theory

This section explores the foundations of spectral theory and the properties of Banach algebras. Here, we begin by defining important algebraic structures, then build up to key results such as Gelfand's spectral radius theorem. We also examine several properties of invertible elements in the spectrum of Banach algebras.

3.1 Invertibility in unital Banach algebras

In Chapter 2 we defined an algebra, which we now extend to the following.

DEFINITION 3.1. A *Banach algebra* is an algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a norm making it a Banach space, such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$.

Moreover, an algebra $\mathcal{A}$ is a *unital algebra* if it possesses a multiplicative identity element. If S is a subset of the Banach algebra $\mathcal{A}$, we say that $\mathcal{A}$ is generated by S if the linear combinations of products of elements of S are dense in $\mathcal{A}$.

EXAMPLE 3.2. ($\mathcal{A} = \ell^1(\mathbb{Z})$ is a commutative unital Banach algebra).

Previously, we have shown that $\mathcal{A} = \ell^1(\mathbb{Z})$ is an algebra. It remains to show that $\mathcal{A}$, with multiplication defined as convolution, is a Banach algebra and contains a multiplicative identity.

To this end, trivially $\mathcal{A} = \ell^1(\mathbb{Z})$ is a Banach space equipped with the norm

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x_n|.$$

Moreover, in showing closure under convolution, we established $\|x * y\|_1 \leq \|x\|_1 \|y\|_1$. Since $\ell^1(\mathbb{Z})$ is a Banach space and convolution satisfies the required algebraic and norm properties, it is a Banach algebra.

To verify that $\delta = \delta^0$ is the unit element, observe that for any $x \in \ell^1(\mathbb{Z})$, $(x * \delta^0)_n = \sum_{k=-\infty}^{\infty} x_k \delta_{n-k} = x_n$, since $\delta_{n-k} = 1$ only when $k = n$ and is zero otherwise. Thus, δ^0 is the multiplicative identity.

Therefore, $\mathcal{A} = \ell^1(\mathbb{Z})$ is a unital Banach algebra with convolution as multiplication and δ^0 as the unit element.

Moreover, convolution is commutative:

$$(x * y)_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k} = \sum_{m=-\infty}^{\infty} x_{n-m} y_m = (y * x)_n.$$

Therefore, $\mathcal{A} = \ell^1(\mathbb{Z})$ is a commutative unital Banach algebra.

Next, we strengthen the definition of a Banach algebra to that of a C*-algebra. To do this, we first introduce the concept of an involution, which is an operation that generalizes the idea of complex conjugation or taking the adjoint of a matrix.

An *involution* on $\mathcal{A}$ is an anti-automorphism of order 2. That is, an involution is a mapping $x \mapsto x^*$ from $\mathcal{A}$ to $\mathcal{A}$ that satisfies:

$$(x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x$$

for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Again, an example of an involution which we will commonly use is complex conjugation.

If an algebra $\mathcal{A}$ is equipped with an involution, $\mathcal{A}$ is called a $*$-*algebra.* Moreover, a Banach $*$-algebra that satisfies $\|x^* x\| = \|x\|^2$ for all $x \in \mathcal{A}$ is called a C^* *algebra.*

Generally, involutions are not required to satisfy, $\|x^*\| = \|x\|$; however, in C^* algebras this property holds. This follows as $\|x\|^2 = \|x^* x\| \leq \|x^*\| \cdot \|x\|$ which implies $\|x\| \leq \|x^*\|$ and conversely, $\|x^*\| \leq \|(x^*)^*\| = \|x^{**}\| = \|x\|$, and so $\|x^*\| = \|x\|$.

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, a *Banach algebra homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi(xy) = \phi(x)\phi(y)$ holds for all $x, y \in \mathcal{A}$. Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are $*-$algebras, a $*-$*homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a homomorphism such that $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{A}$.

The elements of the unital Banach algebra $\mathcal{A}$ that have two-sided inverses are called *invertible elements*. Our next result helps us begin classifying invertible elements of $\mathcal{A}$.

Lemma 3.3. *(Neumann series). Suppose $\mathcal{A}$ is a unital Banach algebra and $x \in \mathcal{A}$. If $\|x\| < 1$, then $e - x$ is invertible, and $(e-x)^{-1} = \sum\limits_{n=0}^{\infty} x^n$.*

Proof. First note, $e - x^{N+1} = (e-x)(\sum_{n=0}^{N} x^n)$, implying $\frac{e-x^{N+1}}{e-x} = \sum_{n=0}^{N} x^n$. Since $\|x\| < 1$, as $N \to \infty$, it follows that $\frac{e-x^{N+1}}{e-x} \to \frac{e}{e-x}$ and $\sum_{n=0}^{N} x^n \to \sum_{n=0}^{\infty} x^n$. Thus, $\sum_{n=0}^{\infty} x^n = \frac{e}{e-x} = (e-x)^{-1}$.

Moreover, $(e-x)(\sum_{n=0}^{N} x^n) = e - x^{N+1} \to e$, and $(\sum_{n=0}^{N} x^n)(e-x) = e - x^{N+1} \to e$ as $N \to \infty$. Therefore $e - x$ is invertible. ■

Similarly, as $(e-x)^{-1} = \sum_0^{\infty} x^n$, it follows that $\|(e-x)^{-1}\| \leq \dfrac{1}{1-\|x\|}$. To show this, consider the partial sum $\sum_{n=0}^{m} x^n$. Then, $\|\sum_{n=0}^{m} x^n\| \leq \sum_{n=0}^{m} \|x\|^n$, and letting $m \to \infty$, we see that

$$\|(e-x)^{-1}\| = \left\| \sum_{n=0}^{\infty} x^n \right\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|}.$$

This result provides a way to understand the invertibility of elements in a Banach algebra, specifically when they lie inside the unit ball of the algebra. We can further generalize this idea to study the invertibility of more general elements, such as those of the form $\lambda e - x$ or $x - y$, in unital Banach algebras.

Theorem 3.4. *Let $\mathcal{A}$ be a unital Banach algebra.*

1. *If $|\lambda| > \|x\|$ then $\lambda e - x$ is invertible, and its inverse is $\sum_0^{\infty} \lambda^{-n-1} x^n$.*

2. *If x is invertible and $\|y\| < \|x^{-1}\|^{-1}$, then $x - y$ is invertible, and*

$$(x-y)^{-1} = x^{-1}\sum_{0}^{\infty}(yx^{-1})^n.$$

3. *If x is invertible and $\|y\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$ then $\|(x-y)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2 \cdot \|y\|$.*

4. *The set of invertible elements of $\mathcal{A}$ is open, and the mapping $x \mapsto x^{-1}$ is continuous on it.*

Proof. To prove the first claim, let $|\lambda| > \|x\|$. Then $\lambda e - x = \lambda(e - \lambda^{-1}x)$ and so $\|\lambda^{-1}x\| = |\lambda^{-1}| \cdot \|x\| = \frac{\|x\|}{|\lambda|} < 1$. Thus by Lemma 3.3, $\lambda e - x$ is invertible with inverse,

$$(\lambda e - x)^{-1} = (\lambda(e - \lambda^{-1}x))^{-1} = \lambda^{-1}(e - \lambda^{-1}x)^{-1} = \lambda^{-1}\left(\sum_{n=0}^{\infty}(\lambda^{-1}x)^n\right)$$
$$= \sum_{n=0}^{\infty}\lambda^{-n-1}x^n.$$

For the second claim, suppose that x is invertible and $\|y\| < \|x^{-1}\|^{-1}$. Then, $x - y = x(e - yx^{-1})$. Moreover, $\|y\| < \|x^{-1}\|^{-1}$ implies $\|y\| \cdot \|x^{-1}\| < 1$ and so $\|yx^{-1}\| \leq \|y\| \cdot \|x^{-1}\| < 1$. Thus, $x - y$ is invertible by Lemma 3.3 and its inverse is given by,

$$(x-y)^{-1} = ((e - yx^{-1})x)^{-1} = x^{-1}(e - yx^{-1})^{-1} = x^{-1}\left(\sum_{n=0}^{\infty}(yx^{-1})^n\right).$$

For the third claim, suppose x is invertible and let $\|y\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$. Then $\|y\| < \|x^{-1}\|^{-1}$ so by the previous, $x-y$ is invertible, and $(x-y)^{-1} = x^{-1}\sum_{n=0}^{\infty}(yx^{-1})^n$. Then,

$$(x-y)^{-1} - x^{-1} = \left(x^{-1}\sum_{n=0}^{\infty}(yx^{-1})^n\right) - x^{-1} = x^{-1}\left(\sum_{n=0}^{\infty}(yx^{-1})^n - 1\right)$$
$$= x^{-1}\sum_{n=1}^{\infty}(yx^{-1})^n.$$

Therefore, we see that

$$\begin{aligned}\|(x-y)^{-1}-x^{-1}\| = \|x^{-1}\sum_{n=1}^{\infty}(yx^{-1})^n\| &\leq \|x^{-1}\|\sum_{n=1}^{\infty}(\|y\|\cdot\|x^{-1}\|)^n \\ &= \|x^{-1}\|\frac{\|y\|\cdot\|x^{-1}\|}{1-\|y\|\cdot\|x^{-1}\|} \\ &\leq \|x^{-1}\|\frac{\|y\|\cdot\|x^{-1}\|}{1-\frac{1}{2}\|x^{-1}\|^{-1}\cdot\|x^{-1}\|} \\ &= 2\|x^{-1}\|^2\cdot\|y\|.\end{aligned}$$

Lastly, to show the fourth claim, denote the set of invertible elements of $\mathcal{A}$ by $\mathrm{Inv}(\mathcal{A})$. To show $\mathrm{Inv}(\mathcal{A})$ is open, let $x \in \mathrm{Inv}(\mathcal{A})$, $y \in \mathcal{A}$ and $h = x - y$. Then for $\|h\| < \|x^{-1}\|^{-1}$, it follows from (2) that

$$x - h = x - (x - y) = y \in \mathrm{Inv}(\mathcal{A}).$$

Thus $b_{\|x^{-1}\|^{-1}}(x) \subseteq \mathrm{Inv}(\mathcal{A})$, and so we conclude that $\mathrm{Inv}(\mathcal{A})$ is open. It remains to show that $x \mapsto x^{-1}$ is continuous on $\mathrm{Inv}(\mathcal{A})$.

To that end, for all $\varepsilon > 0$ let $\delta = \min\{\frac{\varepsilon}{2\|x^{-1}\|^2}, \frac{1}{2}\|x^{-1}\|^{-1}\}$ and consider an arbitrary sequence (x_n) converging to some x on $\mathrm{Inv}(\mathcal{A})$. Define the sequence (y_n) by $y_n = x - x_n$. As $\mathrm{Inv}(\mathcal{A})$ is open, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $b_\delta(x) \subseteq \mathrm{Inv}(\mathcal{A})$. Then whenever $\|y_n\| = \|x - x_n\| < \delta$, by (3) we have

$$\|x_n^{-1} - x^{-1}\| = \|(x-y_n)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\cdot\|y_n\| < 2\|x^{-1}\|^2\cdot\frac{\varepsilon}{2\|x^{-1}\|^2} = \varepsilon.$$

Therefore, $x \mapsto x^{-1}$ is continuous on $\mathrm{Inv}(\mathcal{A})$. ■

3.2 Spectral theory

The last result provides a framework to understand invertibility in a unital Banach algebra, particularly for elements of the form $\lambda e - x$ or $x - y$. This leads to the concept of the spectrum of an element x, which is the set of scalars λ such that $\lambda e - x$ is not invertible.

DEFINITION 3.5. If $x \in \mathcal{A}$, the *spectrum* of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}.$$

Note, by parts 1 and 4 of Theorem 3.4, $\sigma(x)$ is a closed subset of the disk $\{\lambda : |\lambda| \leq \|x\|\}$. Moreover, when we equip an algebra with an involution, additional structure and symmetry arise, which are reflected in the behavior of the spectrum under the involution. This leads us to the following proposition, which establishes key properties of the involution in a unital Banach $*$-algebra.

PROPOSITION 3.6. *Let $\mathcal{A}$ be a unital Banach $*$-algebra.*

1. $e = e^*$.
2. *If x is invertible, then so is x^*, and $(x^*)^{-1} = (x^{-1})^*$.*
3. $\sigma(x^*) = \overline{\sigma(x)}$ *for any $x \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is a unital Banach $*-$algebra, and let $x \in \mathcal{A}$. For the first claim, observe that, $x^* = (xe)^* = e^*x^*$, and so e^* is a multiplicative identity. Thus, $e^* = e$.

For the second claim, suppose $x \in \mathcal{A}$ is invertible. Then there exits $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = e$. Applying the involution to both sides, we see $(x^{-1})^*x^* = (xx^{-1})^* = e^* = e$ and so x^* is invertible. Moreover, $(x^*)^{-1}x^* = e^* = (xx^{-1})^* = (x^{-1})^*x^*$, implies that $(x^*)^{-1} = (x^{-1})^*$.

For the last claim, observe that for $\lambda \in \sigma(x^*)$, we have $\lambda e - x^*$ is not invertible and so $(\lambda e - x^*)^* = \overline{\lambda} e - x$ is not invertible. Thus, $\overline{\lambda} \in \sigma(x)$ implies $\lambda \in \overline{\sigma(x)}$ and so $\sigma(x^*) \subseteq \overline{\sigma(x)}$. For reverse inclusion, the same argument holds with reverse order. Therefore, $\sigma(x^*) = \overline{\sigma(x)}$. ■

To further analyze the spectrum of an element x, it is useful to introduce the concept of the resolvent of x.

DEFINITION 3.7. For $\lambda \notin \sigma(x)$, the *resolvent* of x is the element

$$R(\lambda) = R_x(\lambda) = (\lambda e - x)^{-1}.$$

Using the resolvent, we now aim to show that the spectrum of an element is always nonempty. First, we recall the following definition.

DEFINITION 3.8. Let G be a region of $\mathbb{C}$ and consider $f : G \to \mathbb{C}$. Then f is *analytic* if f is continuously differentiable on G.

That is, f' exists and is also continuous. A function that is defined and analytic in the whole complex plane is said to be an *entire function.*

LEMMA 3.9. *$R(\lambda)$ is an analytic function of $\lambda \in \mathbb{C} \setminus \sigma(x)$.*

Proof. For $\lambda \in C \setminus \sigma(x)$, let $R(\lambda)$ denote the resolvent of x. By Theorem 3.4, $R(\lambda)$ is continuous. To show it is differentiable, suppose $\lambda, \mu \notin \sigma(x)$. Then,

$$\begin{aligned} R'(\lambda) = \lim_{\mu \longrightarrow \lambda} \frac{R(\lambda) - R(\mu)}{\lambda - \mu} &= \lim_{\mu \longrightarrow \lambda} (\frac{1}{\lambda - \mu})(\frac{1}{\lambda e - x} - \frac{1}{\mu e - x}) \\ &= \lim_{\mu \longrightarrow \lambda} (\frac{1}{\lambda - \mu})(\frac{(\mu e - x) - (\lambda e - x)}{(\mu e - x)(\lambda e - x)}) \\ &= \lim_{\mu \longrightarrow \lambda} \frac{e(\mu - \lambda)}{(\lambda - \mu)(\mu e - x)(\lambda e - x)} \\ &= \lim_{\mu \longrightarrow \lambda} -R(\mu)R(\lambda) \\ &= -R(\lambda)^2. \end{aligned}$$

Thus, $R(\lambda)$ is an analytic function of $\mathbb{C} \setminus \sigma(x)$. ∎

This shows that $R(\lambda)$ is an analytic $\mathcal{A}$-valued function on the open set $\mathbb{C} \setminus \sigma(x)$. In particular, this implies that for any bounded linear functional ϕ, $\phi \circ R(\lambda)$ is an ordinary $\mathbb{C}$-valued analytic function of λ. Next, we wish to show $\sigma(\mathcal{A})$ is always nonempty. For this, we need the following theorem from Complex analysis.

THEOREM 3.10 (Liouville's theorem [2], 3.4.). *If f is a bounded entire function, then f is constant.*

Using these, we can now show that the spectrum of any element in a Banach algebra is nonempty by analyzing the behavior of the resolvent function.

PROPOSITION 3.11. *$\sigma(x)$ is nonempty for every $x \in \mathcal{A}$.*

Proof. If $\sigma(x)$ is empty, by Lemma 3.9, $R(\lambda)$ is analytic in all of $\mathbb{C}$ and so it is entire. Additionally,

$$\begin{aligned}
\lim_{\lambda \to \infty} \|R(\lambda)\| = \lim_{\lambda \to \infty} \|(\lambda e - x)^{-1}\| &= \lim_{\lambda \to \infty} |\lambda|^{-1} \cdot \|(e - \lambda^{-1}x)^{-1}\| \\
&= \lim_{\lambda \to \infty} \frac{\|(e - \lambda^{-1}x)^{-1}\|}{|\lambda|} \\
&= 0,
\end{aligned}$$

implying $R(\lambda)$ is also bounded. Thus, by Theorem 3.10 (Liouville's theorem), $R(\lambda)$ would be constant. Moreover, $R(\lambda)$ would be identically zero, a contradiction. Thus, $\sigma(x)$ is nonempty. ∎

This result has great implications for the structure of Banach algebras. In particular, it leads to a fundamental theorem in the theory of Banach algebras: the Gelfand-Mazur theorem. This theorem characterizes Banach algebras in which every nonzero element is invertible, showing that they are isomorphic to $\mathbb{C}$.

Theorem 3.12. *(Gelfand-Mazur theorem).*

If $\mathcal{A}$ is a Banach algebra where every nonzero element is invertible, then $\mathcal{A} \cong \mathbb{C}$.

Proof. Suppose $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible. Consider $\mathbb{C}e = \{\lambda e : \lambda \in \mathbb{C}\}$ and let $x \in \mathcal{A}$. If $x \notin \mathbb{C}e$, then $x \neq \lambda e$ for every $\lambda \in \mathbb{C}$. Then, $\lambda e - x$ is invertible for all $\lambda \in \mathbb{C}$ which implies $\sigma(x)$ is empty, a contradiction to Proposition 3.11.

Therefore, since $x \in \mathbb{C}e$, it follows that $\mathcal{A} \subseteq \mathbb{C}e$, which leads to the conclusion that $\mathcal{A} = \mathbb{C}e$. Moreover, as $\lambda e \mapsto \lambda$ is an isomorphism between $\mathbb{C}e$ and $\mathbb{C}$, it follows that $\mathcal{A} \cong \mathbb{C}$. ■

To further analyze the spectrum of an element in a Banach algebra, we introduce the spectral radius.

Definition 3.13. If $x \in \mathcal{A}$, the *spectral radius* of x is

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Note that the contrapositive of Theorem 3.4, part 1, gives an upper bound for $\rho(x)$. The next result lets us improve on this bound.

Theorem 3.14. *(Gelfand spectral radius theorem).*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Proof. First note, $\lambda^n e - x^n = (\lambda e - x)\sum_{j=0}^{n-1} \lambda^j x^{n-1-j}$ and so if $\lambda^n e - x^n$ is invertible, so is $\lambda e - x$. This implies that if $\lambda e - x$ is not invertible, $\lambda^n e - x^n$ is not invertible. Thus, $\lambda \in \sigma(x)$ implies $\lambda^n \in \sigma(x^n)$. By the contrapositive of Theorem 3.4, for $\lambda^n \in \sigma(x^n)$, we have $|\lambda^n| = |\lambda|^n \leq \|x^n\|$, which implies $|\lambda| \leq \|x^n\|^{1/n}$.

Taking the lim inf of both sides, we have $\liminf_{n\to\infty} |\lambda| \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$ which implies,

$|\lambda| \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$ for any $\lambda \in \sigma(x)$. Thus,

$$\sup\{|\lambda| : \lambda \in \sigma(x)\} = \rho(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}.$$

Conversely, let ϕ be a bounded linear functional in $\mathcal{A}^*$, the dual space of $\mathcal{A}$. Define $T_n(\phi) = \lambda^{-n}\phi(x^n)$, for $\phi \in A^*$, $n \in \mathbb{N}$ and let $A = \{T_n\}_{n\in\mathbb{N}}$ denote this family of operators. Then,

$$[\phi \circ R(\lambda)](x) = \phi(R_x(\lambda)) = \phi((\lambda e - x)^{-1}) \qquad (|\lambda| > \rho(x))$$

is analytic as $\phi(R_x(\lambda))$ is continuous. By Theorem 3.4, the laurent series of $\phi \circ R(\lambda)$ is

$$[\phi \circ R(\lambda)](x) = \phi(R_x(\lambda)) = \phi\left(\sum_{n=0}^{\infty} \lambda^{-n-1}x^n\right) = \sum_{n=0}^{\infty} \lambda^{-n-1}\phi(x^n)$$

which is absolutely convergent for $|\lambda| > \rho(x)$, and is analytic. Then,

$$\sum_{n=0}^{\infty} \lambda^{-n-1}\phi(x^n) = \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-n}\phi(x^n)$$

and so $\lambda^{-1}(\sum_{n=0}^{\infty} |\lambda^{-n}\phi(x^n)|)$ converges. Thus, $\sup_{T\in A} \|T_n(\phi)\| = \sup_{T\in A} \|\lambda^{-n}\phi(x^n)\| < \infty$, for all $\phi \in \mathcal{A}^*$. By Theorem 2.13 (Uniform Boundedness Principle), as $\phi \circ R(\lambda) : \mathcal{A}^* \setminus \sigma(x) \to \mathbb{C}$, we have $\sup_{T\in A} \|\lambda^{-n}x^n\| < \infty$. Thus, there exists some $C < \infty$ such that $\sup_{T\in A} \|\lambda^{-n}x^n\| \leq C$ and so

$$|\lambda|^{-n}\|x^n\| = |\lambda^{-n}|\|x^n\| = \|\lambda^{-n}x^n\| \leq C.$$

Thus, $|\lambda|^{-n}\|x^n\| \leq C$, which implies $\|x^n\| \leq |\lambda|^n C$, and so $\|x^n\|^{1/n} \leq (C|\lambda|^n)^{1/n} = C^{1/n}|\lambda|$. Taking the lim sup of both sides, we see that

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \limsup_{n\to\infty} C^{1/n}|\lambda| = |\lambda| \limsup_{n\to\infty} C^{1/n}$$

Thus, as $\lim_{n\to\infty} C^{1/n} = 1$, we have

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq |\lambda| \leq \rho(x).$$

Therefore, $\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \rho(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$, which implies that,

$$\rho(x) = \lim_{n\longrightarrow\infty} \|x^n\|^{1/n}.$$

Note, to show $\lim_{n\to\infty} C^{1/n} = 1$, let $C = e^{ln(C)}$. Then, $\lim_{n\longrightarrow\infty} C^{1/n} = \lim_{n\longrightarrow\infty} e^{\frac{ln(C)}{n}} = e^0 = 1$. ■

Chapter 4

Gelfand Theory

Multiplicative functionals, maximal ideals, and the spectrum of an algebra are deeply connected. In this section, we see how these interact in the structure of a commutative unital Banach algebra. Specifically, we establish a one-to-one correspondence between the set of multiplicative functionals and the set of maximal ideals. We then establish the Gelfand transform and build up to the Gelfand-Naimark theorem. We lastly examine a specific case of the Gelfand transform, on $L^1(\mathbb{R})$.

4.1 Ideals and the Spectrum

Definition 4.1. Let $\mathcal{A}$ be a commutative unital Banach algebra. A *multiplicative functional* on $\mathcal{A}$ is a nonzero homomorphism from $\mathcal{A}$ to $\mathbb{C}$. The set of all multiplicative functionals on $\mathcal{A}$ is called the *spectrum of* $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$.

We begin this section by exploring the properties of multiplicative functionals. The following establishes characteristics such as their behavior on the identity element, invertible elements, and their relationship to the algebra norm.

Proposition 4.2. *Suppose $h \in \sigma(\mathcal{A})$.*

1. $h(e) = 1$.

2. *If x is invertible in $\mathcal{A}$, then $h(x) \neq 0$.*

3. $|h(x)| \leq \|x\|$ *for all $x \in \mathcal{A}$.*

Proof. Suppose that $h \in \sigma(\mathcal{A})$. For the first claim, let $x \in \mathcal{A}$ such that $h(x) \neq 0$. Then $h(x) = h(ex) = h(e)h(x)$. Thus, it follows that $h(e) = 1$.

For the second claim, suppose that $x \in \mathcal{A}$ is invertible. Then, there exists $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = e$. Since $h(x)h(x^{-1}) = h(xx^{-1}) = h(e) = 1$, it follows that $h(x) \neq 0$.

For the last claim, suppose that $|\lambda| > \|x\|$. Then by Theorem 3.4 $\lambda e - x$ is invertible, which implies that $\lambda - h(x) = h(\lambda e - x) \neq 0$. Taking $\lambda = h(x)$, we see this is a contradiction. Therefore, $|h(x)| \leq \|x\|$. ∎

Part (3) of Proposition 4.2 says that $\sigma(\mathcal{A})$ is a subset of the closed unit ball $\mathcal{B}$ of $\mathcal{A}^*$. Additionally, $\sigma(\mathcal{A})$ is made into a topological space by imposing its weak* topology as a subset of $\mathcal{A}^*$, which is the topology of point-wise convergence on $\mathcal{A}$.

In view of (1) from Proposition 4.2, for an algebra homomorphism $h : \mathcal{A} \to \mathbb{C}$, the conditions $h \neq 0$ and $h(e) = 1$ are equivalent. Thus,

$$\sigma(\mathcal{A}) := \{h \in \mathcal{B} : h(e) = 1 \text{ and } h(xy) = h(x)h(y) \text{ for all } x, y \in \mathcal{A}\}.$$

As $h(e) = 1$ and $h(xy) = h(x)h(y)$ are preserved under point-wise limits, it follows that $\sigma(\mathcal{A})$ is a closed subset of $\mathcal{B}$ in the weak* topology. Thus, by Theorem 2.22 (Banach-Alaoglu theorem), $\sigma(\mathcal{A}) \subseteq \mathcal{B}$ is a compact Hausdorff space.

Next, we begin to explore the relationship between multiplicative functionals and maximal ideals.

Definition 4.3. If $\mathcal{A}$ is any algebra, the *left (right) ideal* of $\mathcal{A}$ is a subalgebra $\mathcal{J}$ of $\mathcal{A}$ such that $xy \in \mathcal{J}$ whenever $x \in \mathcal{A}$ and $y \in \mathcal{J}$ ($x \in \mathcal{J}$ and $y \in \mathcal{A}$). If $\mathcal{A}$ is commutative, it is called an *ideal* instead of a left or right ideal.

Moreover, we say $\mathcal{J}$ is a proper ideal of $\mathcal{A}$ if $\mathcal{J} \neq \mathcal{A}$. If $\mathcal{A}$ is a unital Algebra, then $\mathcal{J}$ is proper if and only if $e \notin \mathcal{J}$. This follows as if $e \in \mathcal{J}$, then for any $x \in \mathcal{A}$

we have $x = ex = xe \in \mathcal{A}$, implying $\mathcal{A} = \mathcal{J}$. Lastly, we say $\mathcal{J}$ is a *maximal ideal* if it is a proper ideal that is not contained in any larger proper ideal.

Ideals are key to understanding the structure of a Banach algebra. The next proposition establishes several properties of ideals such as their relationship to invertibility.

PROPOSITION 4.4. *Let $\mathcal{A}$ be a commutative unital Banach algebra, and $\mathcal{J} \subset \mathcal{A}$ be a proper ideal.*

1. *$\mathcal{J}$ contains no invertible elements.*

2. *The closure of $\mathcal{J}$, denoted $\overline{\mathcal{J}}$, is a proper ideal.*

3. *$\mathcal{J}$ is contained in a maximal ideal.*

4. *If $\mathcal{J}$ is a maximal ideal, then $\mathcal{J}$ is closed.*

Proof. To prove the first, let $\mathcal{J}$ be a proper ideal of $\mathcal{A}$ and suppose that $x \in \mathcal{J}$ is invertible. Then there exists some $x^{-1} \in \mathcal{A}$ such that $x \cdot x^{-1} = e \in \mathcal{J}$. Thus, for any $y \in \mathcal{A}$, we have $e \cdot y = y \in \mathcal{J}$, so $\mathcal{A} \subseteq \mathcal{J}$. Therefore, $\mathcal{J} = \mathcal{A}$, but since $\mathcal{J}$ is a proper ideal, it follows that $\mathcal{J}$ cannot contain any invertible elements.

Now for the second, if $\mathcal{J}$ is a proper ideal of $\mathcal{A}$, then from (1), it contains no invertible elements. Thus, $\mathcal{J}$ is contained in the set of non-invertible elements of $\mathcal{A}$. By part 4 of Theorem 3.4, the set of invertible elements is open, which implies that the set of non-invertible elements is closed. Therefore, $\overline{\mathcal{J}}$ is also contained in the set of non-invertible elements, so $e \notin \overline{\mathcal{J}}$.

To show that $\overline{\mathcal{J}}$ is an ideal, let $x \in \overline{\mathcal{J}}$ and $y \in \mathcal{A}$. Since $\overline{\mathcal{J}}$ is closed, there exists a sequence $(x_n) \in \mathcal{J}$ such that $x_n \to x$ as $n \to \infty$. As $\mathcal{J}$ is an ideal, $x_n \cdot y \in \mathcal{J}$ for all n. Moreover, since multiplication is continuous, $x_n \cdot y \to x \cdot y$, so $x \cdot y \in \overline{\mathcal{J}}$. Therefore, $\overline{\mathcal{J}}$ is a proper ideal of $\mathcal{A}$.

For the third, let $\mathcal{S}$ be the set of all proper ideals that contain $\mathcal{J}$. That is,

$$\mathcal{S} := \{\mathcal{I} \subset \mathcal{A} : \mathcal{I} \text{ is a proper ideal, and } \mathcal{J} \subseteq \mathcal{I}\}.$$

Note that $\mathcal{J} \subset \mathcal{A}$ and $\mathcal{J} \subseteq \mathcal{J}$, so $\mathcal{S}$ is nonempty.

Next, define a partial ordering on $\mathcal{S}$ by $I_1 \leq I_2$ whenever $I_1 \subseteq I_2$. Suppose $\{I_\alpha\}_{\alpha \in A}$ is a chain in $\mathcal{S}$, and define $\mathcal{I} := \bigcup_{\alpha \in A} I_\alpha$. We claim that $\mathcal{I} \in \mathcal{S}$ is an upper bound for $\{I_\alpha\}_{\alpha \in A}$.

Observe that $\mathcal{I} \in \mathcal{S}$ is an upper bound, as for any $\alpha_0 \in A$, we have $I_{\alpha_0} \subseteq \bigcup_{\alpha \in A} I_\alpha = \mathcal{I}$, and so $I_{\alpha_0} \leq \mathcal{I}$. It remains to show that $\mathcal{I} \in \mathcal{S}$.

To this end, for any $\alpha \in A$, note that $\mathcal{J} \subseteq \mathcal{I}_\alpha$, so $\mathcal{J} \subset \bigcup_{\alpha \in A} I_\alpha$. To see that $\mathcal{I}$ is proper, for any $\alpha \in A$ observe that $e \notin I_\alpha$, as each I_α is proper. Thus, $e \notin \bigcup_{\alpha \in A} I_\alpha = \mathcal{I}$. Lastly, to show $\mathcal{I}$ is an ideal, suppose $x \in \mathcal{I}$ and $y \in \mathcal{A}$. Then there exists some $\alpha_0 \in A$ such that $x \in I_{\alpha_0}$. As I_{α_0} is an ideal, it follows that $x \cdot y \in I_{\alpha_0}$, and so $x \cdot y \in \bigcup_{\alpha \in A} I_\alpha = \mathcal{I}$.

Thus, every chain $\{I_\alpha\}_{\alpha \in A}$ in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. By Lemma 2.3 (Zorn's lemma), $\mathcal{S}$ has a maximal element, call it $\mathcal{M}$. Therefore, as $\mathcal{J} \subset \mathcal{M}$, it follows $\mathcal{J}$ is contained in a maximal ideal.

Lastly, for the fourth, suppose $\mathcal{J}$ is a maximal ideal. From (2), $\overline{\mathcal{J}}$ is a proper ideal, which contains $\mathcal{J}$. Thus, $\mathcal{J}$ being maximal implies that $\mathcal{J} = \overline{\mathcal{J}}$ and so $\mathcal{J}$ is closed. ∎

Next, we recall a fundamental result about the quotients of Banach algebras.

PROPOSITION 4.5. *If $\mathcal{A}$ is a Banach algebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ is a Banach algebra.*

Proof. Suppose $\mathcal{A}$ is a Banach algebra, and $\mathcal{J} \subseteq \mathcal{A}$ is a closed ideal. To show that $\mathcal{A}/\mathcal{J}$ is an algebra, define multiplication on $\mathcal{A}/\mathcal{J}$ by

$$[a] \cdot [b] := [a \cdot b],$$

where $[a] = a + \mathcal{J}$ and $[b] = b + \mathcal{J}$.

To show this is well defined, suppose $[a_1] = [a_2]$ and $[b_1] = [b_2]$. Then, $a_1 - a_2 \in \mathcal{J}$ and $b_1 - b_2 \in \mathcal{J}$, so

$$a_1 b_1 - a_2 b_2 = a_1(b_1 - b_2) + (a_1 - a_2)b_2 \in \mathcal{J},$$

as $\mathcal{J}$ is an ideal. Thus, $a_1b_1 - a_2b_2 \in \mathcal{J}$ implies that $[a_1b_1] = [a_2b_2]$, and so multiplication is well-defined. Therefore, $\mathcal{A}/\mathcal{J}$ inherits the algebraic structure from $\mathcal{A}$, and so $\mathcal{A}/\mathcal{J}$ is an algebra.

To see that $\mathcal{A}/\mathcal{J}$ is a Banach algebra, define the norm on $\mathcal{A}/\mathcal{J}$ to be the quotient norm:

$$\|[a]\|_{\mathcal{A}/\mathcal{J}} := \inf_{j \in \mathcal{J}} \|a + j\|_{\mathcal{A}}.$$

First, we will show this satisfies the norm inequality for a Banach algebra. To this end, consider arbitrary $a, b \in \mathcal{A}$ and $j_1, j_2 \in \mathcal{J}$. Then, for all $\varepsilon > 0$, there exists $j_1, j_2 \in \mathcal{J}$ such that $\|a + j_1\|_A \leq \|[a]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon$ and $\|b + j_2\|_A \leq \|[b]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon$. Next, for $ab + j \in \mathcal{A} \setminus \mathcal{J}$, take $j = aj_2 + j_1 b + j_1 j_2$, which is clearly in $\mathcal{J}$. Then

$$\begin{aligned}\|ab + j\|_{\mathcal{A}} = \|ab + (aj_2 + j_1 b + j_1 j_2)\|_{\mathcal{A}} &= \|(a + j_1)(b + j_2)\|_{\mathcal{A}} \\ &\leq \|a + j_1\|_{\mathcal{A}} \|b + j_1\|_{\mathcal{A}} \\ &\leq (\|[a]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon)(\|[b]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon).\end{aligned}$$

Since $\|[ab]|_{\mathcal{A}\setminus\mathcal{J}}$ is the infimum over all $j \in \mathcal{J}$, we have

$$\|[ab]|_{\mathcal{A}\setminus\mathcal{J}} \leq \|ab + j\|_{\mathcal{A}} \leq (\|[a]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon)(\|[b]\|_{\mathcal{A}\setminus\mathcal{J}} + \varepsilon),$$

for all $\varepsilon > 0$. Thus, as $\varepsilon \to 0$, we have

$$\|[ab]|_{\mathcal{A}\setminus\mathcal{J}} \leq \|[a]\|_{\mathcal{A}\setminus\mathcal{J}} \|[b]\|_{\mathcal{A}\setminus\mathcal{J}}.$$

Lastly, we show that $\mathcal{A}/\mathcal{J}$ is complete. To that end, consider a Cauchy sequence of equivalence classes $([a_n])_{n\in\mathbb{N}}$ in $\mathcal{A}/\mathcal{J}$. Then, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|[a_n] - [a_m]\|_{\mathcal{A}/\mathcal{J}} = \inf_{j\in\mathcal{J}} \|(a_n - a_m) + (j_n - j_m)\|_{\mathcal{A}} < \varepsilon, \tag{$*$}$$

which implies that

$$\inf_{j\in\mathcal{J}} \|a_n - a_m\|_{\mathcal{A}} + \inf_{j\in\mathcal{J}} \|j_n - j_m\|_{\mathcal{A}} \leq \inf_{j\in\mathcal{J}} \|(a_n - a_m) + (j_n - j_m)\|_{\mathcal{A}} < \varepsilon.$$

Taking $j_n = j_m$, we see that $\inf\limits_{j\in\mathcal{J}} \|a_n - a_m\|_{\mathcal{A}} = \|a_n - a_m\|_{\mathcal{A}} < \varepsilon$, and so $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. As $\mathcal{A}$ is a Banach algebra, it is complete and so there exists an $a \in \mathcal{A}$ such that $a_n \to a$ as $n \to \infty$. As $a + \mathcal{J}$ is in $\mathcal{A}/\mathcal{J}$, it remains to show that $[a_n] \to [a]$.

To that end, from $(*)$, there exists an $n \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$\|[a_n] - [a_m]\|_{\mathcal{A}/\mathcal{J}} = \inf_{j\in\mathcal{J}} \|(a_n - a_m) + j\|_{\mathcal{A}} < \frac{\varepsilon}{2}.$$

Taking the limit as $m \to \infty$, we see that

$$\lim_{m\to\infty} \inf_{j\in\mathcal{J}} \|(a_n - a_m) + j\|_{\mathcal{A}} = \|(a_n - a) + j\| \leq \frac{\varepsilon}{2}.$$

Thus, $\|[a_n] - [a]\|_{\mathcal{A}/\mathcal{J}} \leq \frac{\varepsilon}{2} < \epsilon$, and so $([a_n])_{n\in]\mathbb{N}}$ converges to $[a]$ in $\mathcal{A}/\mathcal{J}$. Therefore, $\mathcal{A}/\mathcal{J}$ is a Banach algebra. ■

The construction of quotient Banach algebras from closed ideals provides a tool for analyzing the original algebra. Our next major result establishes a one-to-one correspondence between the set of maximal ideals and the spectrum of a commutative Banach algebra. To build towards this, we need the following result and definition.

Definition 4.6. If W is a subspace of a finite-dimensional vector space V, then the *co-dimension* of W in V is $dim(V) - dim(W)$.

Proposition 4.7. *If $\mathcal{A}$ is a commutative Banach algebra, then an ideal is maximal if and only if it has co-dimension one.*

Proof. First, suppose $\mathcal{I}$ is a maximal ideal of $\mathcal{A}$. Note that $\mathcal{I}$ is a closed ideal by Proposition 4.4 and so $\mathcal{A}/\mathcal{I}$ is a field by Proposition 2.8. Moreover, as $\mathcal{A}$ is a Banach algebra, it follows that $\mathcal{A}/\mathcal{I}$ is a Banach algebra by Proposition 4.5. Thus, by Theorem 3.12 (Gelfand-Mazur theorem), it follows that $\mathcal{A}/\mathcal{I} \cong \mathbb{C}$. Therefore, $\dim(\mathcal{A}/\mathcal{I}) = 1$ and so the codimension of $\mathcal{I}$ is 1.

Conversely, suppose $\mathcal{I}$ is an ideal of $\mathcal{A}$ such that $\mathcal{I}$ has codimension 1. Since $\mathcal{A}/\mathcal{I}$ is a one-dimensional vector space over $\mathbb{C}$, it follows that $\mathcal{A}/\mathcal{I} \cong \mathbb{C}$. Therefore $\mathcal{A}/\mathcal{I}$ is a field, and we conclude that $\mathcal{I}$ is a maximal ideal by Proposition 2.8. ■

The relationship highlighted in the previous leads us the the main result of this section: a one-to-one correspondence between the spectrum and the set of maximal ideals in a commutative unital Banach algebra $\mathcal{A}$.

Theorem 4.8. *Let $\mathcal{A}$ be a commutative unital Banach algebra. Then the map $h \mapsto \ker(h)$ is a one-to-one correspondence between $\sigma(\mathcal{A})$ and the set of maximal ideals in $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a commutative unital Banach algebra. If $h \in \sigma(\mathcal{A})$, then $\ker(h)$ is an ideal as for any $x \in \ker(h)$ and $y \in \mathcal{A}$ we have,

$$h(xy) = h(x)h(y) = 0 \cdot h(y) = 0,$$

implying $xy \in \ker(h)$. Additionally, $\ker(h)$ is also a proper ideal as $h(e) = 1$ and so $e \notin \ker(h)$.

Since h is a multiplicative functional on $\mathcal{A}$, it is a nonzero homomorphism from $\mathcal{A}$ to $\mathbb{C}$, so consider $\operatorname{Im}(h) \subset \mathbb{C}$. As $\mathbb{C}$ is one-dimensional, the only subspaces of $\mathbb{C}$ are the trivial subspace or $\mathbb{C}$ itself. As h is not the zero map, it follows that $\operatorname{Im}(h) = \mathbb{C}$. Thus h is surjective.

By Theorem 2.6 (First Isomorphism theorem), $\tilde{h} : \mathcal{A}/\ker(h) \to \mathbb{C}$ is a well-defined algebra homomorphism and $\tilde{h}$ is injective. Moreover, as h is surjective, $\tilde{h}$ induces an isomorphism between $\mathcal{A}/\ker(h)$ and $\mathbb{C}$. Thus, the codimesion of $\ker(h)$ is $\dim(\mathcal{A}/\ker(h)) = \dim(\mathbb{C}) = 1$. Therefore, by Proposition 4.7, $\ker(h)$ is a maximal ideal of $\mathcal{A}$.

Next, let $g, h \in \sigma(\mathcal{A})$ such that $\ker(g) = \ker(h)$. Define $f : \mathcal{A} \to \mathbb{C}$ such that $f(x) = h(x) - g(x)$, which is clearly a homomorphism. Suppose $x \in \ker(f)$. Then $f(x) = h(x) - g(x) = 0$ and so $h(x) = g(x)$. Thus, the mapping $h \mapsto \ker(h)$ is an injection from $\sigma(\mathcal{A})$ to the set of maximal ideals.

On the other hand, let $\mathcal{M}$ be a maximal ideal of $\mathcal{A}$ and consider the standard quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{M}$. As $\mathcal{M}$ is a maximal ideal, it is closed by Proposition 4.4. Thus, $\mathcal{A}/\mathcal{M}$ is a Banach algebra by Proposition 4.5, with the quotient norm:

$$\|x + \mathcal{M}\|_{\mathcal{A}/\mathcal{M}} = \inf_{m \in \mathcal{M}} \|x + m\|.$$

Additionally, as $\mathcal{M}$ is maximal, by Proposition 2.8, $\mathcal{A}/\mathcal{M}$ is a field, and so every nonzero element is invertible. Then, by Theorem 3.12 (Gelfand-Mazur theorem), it follows that $\mathcal{A}/\mathcal{M} \cong \mathbb{C}$. Let ϕ denote this isomorphism. Then $\phi \circ \pi : \mathcal{A} \to \mathbb{C}$ is a nonzero multiplicative linear functional on $\mathcal{A}$ whose kernel is $\mathcal{M}$.

Therefore, $h \mapsto \ker(h)$ is a one-to-one correspondence between $\sigma(\mathcal{A})$ and the set of maximal ideals. ■

4.2 The Gelfand transform

For an element $x \in \mathcal{A}$, we begin by defining the function $\widehat{x}$ on $\sigma(\mathcal{A})$ as

$$\widehat{x}(h) = h(x).$$

Observe that $\widehat{x}$ is continuous on $\sigma(\mathcal{A})$, as the topology on $\sigma(\mathcal{A})$ is the weak* topology: the topology of point-wise convergence on $\mathcal{A}$.

Definition 4.9. The map $x \mapsto \widehat{x}$ from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$ is called the *Gelfand transform* on $\mathcal{A}$ which is denoted by $\Gamma_{\mathcal{A}}$ or simply Γ :

$$\Gamma(x) = \Gamma_{\mathcal{A}}(x) = \widehat{x}.$$

Theorem 4.10. *Suppose $\mathcal{A}$ is a commutative unital Banach algebra and $x \in \mathcal{A}$.*

1. *The Gelfand transform is a homomorphism from $\mathcal{A}$ to the set of continuous functions on the spectrum of $\mathcal{A}$, $C(\sigma(\mathcal{A}))$. Moreover, $\widehat{e}$ is the constant function 1.*

2. *x is invertible if and only if $\widehat{x}$ never vanishes.*

3. *range$(\widehat{x}) = \sigma(x)$.*

4. $\|\widehat{x}\|_{sup} = \rho(x) \leq \|x\|$

Proof. Let $\mathcal{A}$ is a commutative unital Banach algebra. To prove the first, consider the Gelfand transform $\Gamma_{\mathcal{A}}$. Then for any $x, y \in \mathcal{A}$, and $h \in \sigma(\mathcal{A})$, we have, $\widehat{x+y}(h) = h(x+y) = h(x) + h(y) = \widehat{x}(h) + \widehat{y}(h)$. Similarly, we have $\widehat{x \cdot y}(h) = h(x \cdot y) = h(x) \cdot h(y) = \widehat{x}(h) \cdot \widehat{y}(h)$, and for any $\lambda \in \mathbb{C}$, $\widehat{\lambda x}(h) = h(\lambda x) = \lambda h(x) = \lambda \widehat{x}(h)$. Moreover, as $h \in \sigma(\mathcal{A})$ and $\widehat{x}$ is continuous on $\sigma(\mathcal{A})$, it follows that $\Gamma_{\mathcal{A}}$ is a homomorphism mapping $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$. Moreover, $\widehat{e}(h) = h(e) = 1$.

For the second, suppose $x \in \mathcal{A}$. Observe that if x is not invertible, then by Proposition 4.4, the ideal generated by x is proper. Further, x is contained in a maximal ideal by the same proposition. Thus, by Theorem 4.8, we have $h(x) = 0$ for some $h \in \sigma(\mathcal{A})$. Therefore, $\widehat{x}(h) = h(x)$ vanishes for some $h \in \sigma(\mathcal{A})$. For the other direction, the same argument holds but with the order reversed. Therefore, x is invertible if and only if $\widehat{x}$ never vanishes.

For the third, suppose $f(x) \in \text{range}(\widehat{x})$ and take $\lambda = f(x)$. Then,

$$f(\lambda e - x) = f(\lambda e) - f(x) = \lambda f(e) - f(x) = \lambda - f(x) = f(x) - f(x) = 0.$$

Thus, by (2), $\lambda e - x$ is not invertible and so $\lambda \in \sigma(x)$.

Conversely, suppose that $x \in \mathcal{A}$ and $\lambda \in \sigma(x)$. Then $\lambda e - x$ is not invertible, and so by (2), there exists some $g \in \sigma(x)$ such that $g(\lambda e - x) = 0$. Then, $0 = g(\lambda e - x) = g(\lambda e) - g(x) = \lambda - g(x)$ which implies $g(x) = \lambda$ and so $\lambda \in \text{range}(\widehat{x})$. Therefore, we conclude that $\text{range}(\widehat{x}) = \sigma(x)$.

To prove the last claim, by the previous we have,

$$\begin{aligned} \|\widehat{x}\|_{\text{sup}} &= \sup\{|\widehat{x}(h)| : h \in \sigma(A), x \in \mathcal{A}\} \\ &= \sup\{|h(x)| : h(x) \in \text{range}(\widehat{x})\} \\ &= \sup\{|\lambda| : \lambda = h(x), \lambda \in \sigma(x)\} \\ &= \rho(x). \end{aligned}$$

Additionally, by Theorem 3.4, $\rho(x) \leq \|x\|$. Therefore, $\|\widehat{x}\|_{\text{sup}} = \rho(x) \leq \|x\|$. ■

Theorem 4.10 establishes key properties of the Gelfand transform, including its role as a homomorphism and its connection to the spectrum of elements in a commutative unital Banach algebra. These results highlight the connection of the Gelfand transform between the structure of $\mathcal{A}$ to the structure of $C(\sigma(\mathcal{A}))$.

When $\mathcal{A}$ is equipped with an involution, additional symmetry arises. Specifically, where the Gelfand transform interacts with the involution.

DEFINITION 4.11. A $*$-algebra $\mathcal{A}$ is called *symmetric* if $\widehat{x^*} = \overline{\widehat{x}}$.

That is, the Gelfand transform takes the involution on $\mathcal{A}$ to the canonical involution (complex conjugation) on $C(\sigma(\mathcal{A}))$.

Before our next result, recall the following definition and theorem.

DEFINITION 4.12. A *subalgebra* $\mathcal{B}$ of an algebra $\mathcal{A}$ is a subset $\mathcal{B} \subseteq \mathcal{A}$ that is itself an algebra under the same operations.

THEOREM 4.13 (Stone-Weierstrass theorem [5], 4.45.). , *Suppose X is a compact Hausdorff space and $\mathcal{A}$ is a subalgebra of $C(X)$. If $\mathcal{A}$ separates points, contains the constant function, and is closed under conjugation, then $\mathcal{A}$ is dense in $C(X)$.*

The Stone-Weierstrass theorem proves to be a powerful tool, as it will help us establish the density of $\Gamma_{\mathcal{A}}$ in $C(\sigma(\mathcal{A}))$, as we will see in the following.

PROPOSITION 4.14. *Suppose $\mathcal{A}$ is a commutative unital Banach $*$-algebra.*

1. *$\mathcal{A}$ is symmetric if and only if $\widehat{x}$ is real-valued whenever $x = x^*$.*
2. *If $\mathcal{A}$ is a C^* algebra, $\mathcal{A}$ is symmetric.*
3. *If $\mathcal{A}$ is symmetric, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$.*

Proof. To prove the first, first suppose that $\mathcal{A}$ is symmetric and that $x = x^*$. Since $\mathcal{A}$ is symmetric, we have $\widehat{x^*} = \overline{\widehat{x}}$. Then, $\widehat{x} = \widehat{x^*} = \overline{\widehat{x}}$, and so $\widehat{x}$ is real-valued.

Conversely, suppose that $x \in \mathcal{A}$, $h \in \sigma(\mathcal{A})$ and $\widehat{x}$ is real-valued whenever $x = \widehat{x}$. We decompose x into its symmetric and anti-symmetric parts: $x = u + iv$. Define $u = \frac{x+x^*}{2}$ and $v = \frac{x-x^*}{2i}$. Then $u = u^*$ and $v = v^*$, which follows as,

$$u^* = \left(\frac{x + x^*}{2}\right)^* = \frac{x^* + x^{**}}{2^*} = \frac{x^* + x}{2} = \frac{x + x^*}{2} = u,$$

and

$$v^* = \left(\frac{x - x^*}{2i}\right)^* = \frac{x^* - x^{**}}{(2i)^*} = \frac{x^* - x}{-2i} = \frac{x - x^*}{2i} = v.$$

Since $u = u^*$ and $v = v^*$, $\widehat{u}$ and $\widehat{v}$ are real-valued. Moreover, $x = u + iv$, and $x^* = (u + iv)^* = u^* - iv^* = u - iv$, and so

$$\widehat{x^*}(h) = h(x^*) = h(u - iv) = h(u) - ih(v) = \overline{h(u) + ih(v)} = \overline{h(u + iv)} = \overline{h(x)} = \overline{\widehat{x}(h)}.$$

Therefore, $\widehat{x^*} = \overline{\widehat{x}}$, holds for all $x \in \mathcal{A}$, and so $\mathcal{A}$ is symmetric.

To prove the second, suppose $\mathcal{A}$ is a C^* algebra, and $x = x^* \in \mathcal{A}$. Let $h \in \sigma(\mathcal{A})$ such that $\widehat{x}(h) = h(x) = \alpha + i\beta$, for some real-valued α and β. For some $t \in \mathbb{R}$, consider $z = x + ite$. Then we have,

$$h(z) = h(x + ite) = h(x) + ith(e) = (\alpha + i\beta) + it = \alpha + i(\beta + t),$$

and

$$z^*z = (x + ite)^*(x + ite) = (x^* + e^*t^*i^*)(x + ite) = (x - ite)(x + ite) = x^2 + t^2e.$$

Thus,

$$\begin{aligned} \alpha^2 + (\beta + t)^2 = \|\alpha + i(\beta + t)\|^2 = \|h(z)\|^2 \leq \|z\|^2 = \|z^*z\| = \|x^2 + t^2e\| \\ \leq \|x^2\| + t^2, \end{aligned}$$

where the first inequality follows by Proposition 4.2. Then $\alpha^2+\beta^2+2\beta t+t^2 \leq \|x^2\|+t^2$ implies that $\alpha^2 + \beta^2 + 2\beta t \leq \|x^2\|$ for all $t \in \mathbb{R}$. This forces $2\beta t = 0$ and so $\beta = 0$. But then, $h(x) = \alpha + i\beta = \alpha$, so $h(x) = \widehat{x}(h)$ is real. Therefore, by (2), we conclude that $\mathcal{A}$ is symmetric.

To prove the third, we first show $\Gamma_{\mathcal{A}}$ is a subalgebra of $C(\sigma(\mathcal{A}))$. For this, we must show it is closed under the operations of $C(\sigma(\mathcal{A}))$. To that end, let $\widehat{x}, \widehat{y} \in C(\sigma(\mathcal{A}))$ and $h \in \sigma(\mathcal{A})$. First, for addition on $\Gamma_{\mathcal{A}}$ we have,

$$(\widehat{x} + \widehat{y})(h) = \widehat{x}(h) + \widehat{y}(h) = h(x) + h(y) = h(x + y) = \widehat{(x + y)}(h).$$

Similarly for multiplication,

$$(\widehat{x} \cdot \widehat{y})(h) = \widehat{x}(h) \cdot \widehat{y}(h) = h(x) \cdot h(y) = h(x \cdot y) = \widehat{(x \cdot y)}(h).$$

Finally, for scalar multiplication, let $\lambda \in \mathbb{C}$. Then,

$$\widehat{\lambda x}(h) = h(\lambda x) = \lambda h(x) = \lambda \widehat{x}(h).$$

Since $\Gamma_{\mathcal{A}} \subseteq C(\sigma(\mathcal{A}))$, $\Gamma_{\mathcal{A}}$ inherits the algebraic structure of $C(\sigma(\mathcal{A}))$. Therefore, $\Gamma_{\mathcal{A}}$ is a subalgebra of $C(\sigma(\mathcal{A}))$.

Moreover, since $\mathcal{A}$ is symmetric, from the previous we have $\widehat{x}$ is real-valued whenever $x = x^*$. Therefore, $\Gamma_{\mathcal{A}}$ is closed under complex conjugation. By Theorem 4.10, $\widehat{e}$ is the constant function 1 and so $\Gamma_{\mathcal{A}}$ contains the constant function. Lastly, for any $h_1, h_2 \in \sigma(\mathcal{A})$, there exists an $a \in \mathcal{A}$ such that $h_1(a) \neq h_2(a)$. Thus, $\widehat{a}(h_1) \neq \widehat{a}(h_2)$, and so $\Gamma_{\mathcal{A}}$ separates points on $\sigma(\mathcal{A})$.

Therefore, by the Theorem 4.13 (Stone-Weierstrass theorem), as $\sigma(\mathcal{A})$ is a compact Hausdorff space, it follows that $\Gamma_{\mathcal{A}}$ is dense in $C(\sigma(\mathcal{A}))$. ■

Our next major result establishes a homeomorphism between the spectrum of an element and the spectrum of an algebra. For this, we need the following.

LEMMA 4.15. *Let X and Y be compact Hausdorff spaces. If $f : X \to Y$ is a continuous and injective map, then f is a homeomorphism from X onto its image $f(X)$.*

Proof. Suppose X and Y are compact Hausdorff spaces and let f be a continuous injection. Since the image of a compact set under a continuous map is compact, it follows that $f(X)$ is compact in Y. Moreover, as Y is Hausdorff and $f(X) \subseteq Y$, it follows that $f(X)$ is closed in Y. Therefore, by Theorem 2.24, f is a homeomorphism onto its image. ∎

Proposition 4.16. *If $x_0 \in \mathcal{A}$, $\widehat{x_0}$ is a homeomorphism from $\sigma(\mathcal{A})$ to $\sigma(x_0)$ in each of the following cases:*

1. *$\mathcal{A}$ is generated by x_0 and e;*

2. *x_0 is invertible and $\mathcal{A}$ is generated by x_0 and x_0^{-1};*

3. *$\mathcal{A}$ is symmetric and $\mathcal{A}$ is generated by x_0, x_0^*, and e.*

Proof. Suppose $x_0 \in \mathcal{A}$. By Theorem 4.10 range$(\widehat{x_0}) = \sigma(x_0)$ and so $\widehat{x_0}$ maps $\sigma(\mathcal{A})$ to $\sigma(x_0)$. As $\widehat{x_0}$ is continuous, and both $\sigma(\mathcal{A})$ and $\sigma(x_0)$ are compact Hausdorff spaces, by Lemma 4.15 it suffices to show that $\widehat{x_0}$ is injective. To that end, observe that in each of the three following cases, any $h \in \sigma(\mathcal{A})$ is completely determined by its action on x_0.

1. If $\mathcal{A}$ is generated by x_0 and e, this is clear.

2. Let x_0 be invertible and suppose that $\mathcal{A}$ is generated by x_0 and x_0^{-1}. As $h \in \sigma(\mathcal{A})$ is a multiplicative functional, we have,

$$\widehat{x_0^{-1}}(h) = h(x_0^{-1}) = h(x_0)^{-1} = \widehat{x_0}^{-1}(h).$$

3. Suppose $\mathcal{A}$ is symmetric, and that its generated by x_0, x_0^*, and e. Then,

$$\widehat{x_0^*}(h) = h(x_0^*) = \overline{h(x_0)} = \overline{\widehat{x_0}(h)}.$$

So, suppose that $h_1, h_2 \in \sigma(\mathcal{A})$. As any $h \in \sigma(\mathcal{A})$ is completely determined by its action on x_0, it follows that if $\widehat{x_0}(h_1) = \widehat{x_0}(h_2)$, then $h_1 = h_2$. Therefore, $\widehat{x_0}$ is an injection, and so by Lemma 4.15, $\widehat{x_0}$ is a homeomorphism from $\sigma(\mathcal{A})$ to $\sigma(x_0)$ in all three cases. ■

This result is the motivation for calling $\sigma(\mathcal{A})$ the *spectrum* of $\mathcal{A}$. We now examine how the Gelfand transform establishes a homeomorphism from a compact Hausdorff space X, to the spectrum of the set of continuous functions on X.

THEOREM 4.17. *Let X be a compact Hausdorff space. For each $x \in X$, define $h_x : C(X) \to \mathbb{C}$ by $h_x(f) = f(x)$. Then the map $x \mapsto h_x$ is a homeomorphism from X to $\sigma(C(X))$. If we identify $x \in X$ with $h_x \in \sigma(C(X))$, the Gelfand transform on $C(X)$ becomes the identity map.*

Proof. Suppose X is a compact Hausdorff space, and for each $x \in X$, define $h_x : C(X) \to \mathbb{C}$ by $h_x(f) = f(x)$. Consider $f, g \in C(X)$. Then for each $x \in X$,

$$h_x(f \cdot g) = (f \cdot g)(x) = f(x) \cdot g(x) = h_x(f) \cdot h_x(g),$$

and so h_x is a multiplicative linear functional on $C(X)$.

Moreover, as X is Hausdorff, for all $x, y \in X$ there exist disjoint neighborhoods of x and y. That is, there exists closed sets A and B, and open sets U, and V such that $x \in U \subset A$ and $y \in V \subset B$ which implies that X is normal. By Lemma 2.28 (Urysohn's lemma), there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Thus, continuous functions separate points on X, implying $h_x \neq h_y$ for $x \neq y$.

Next, let $(x_\alpha)_{\alpha \in A}$ be a net in X such that $x_\alpha \to x$. Then for each $f \in C(X)$, we have $f(x_\alpha) \to f(x)$ and so $h_{x_\alpha} \to h_x$ in the weak* topology on $\sigma(C(X))$. Since these spaces are compact and Hausdorff, $x \mapsto h_x$ is a homeomorphism onto its image by Lemma 4.15. Thus, it remains to show that every multiplicative functional on $C(X)$ is of the form h_x for some $x \in X$.

By Theorem 4.8, there is a one-to-one correspondence between $\sigma(C(X))$ and the set of maximal ideals, by $f \mapsto \ker(f)$. Thus, it is equivalent to show that every maximal ideal in $C(X)$ is of the form $\mathcal{M}_x = \{f : f(x) = 0\}$, for some $x \in X$. This is equivalent to showing, by Proposition 4.4, that every proper ideal $\mathcal{J} \subset C(X)$ is contained in some $\mathcal{M}_x$.

Suppose to the contrary, for each $x \in X$ that there is some $f_x \in \mathcal{J}$ such that $f_x(x) \neq 0$. Then the open sets $\{y : f_x(y) \neq 0\}$ form an open cover for X. As X is compact, $\{y : f_x(y) \neq 0\}$ has a finite subcover $f_1, f_2, \ldots, f_n \in \mathcal{J}$, which have no common zeros. Define $g = \sum_{i=1}^{n} |f_i|^2$. Then $g = \sum_{i=1}^{n} f_i \cdot f_i \in \mathcal{J}$ as $\mathcal{J}$ is an ideal. Moreover, as $g > 0$ everywhere, g is invertible in $C(X)$. By Proposition 4.4, as $\mathcal{J}$ is a proper ideal, it contains no invertible elements, a contradiction. Thus, $\mathcal{J} \subset \mathcal{M}_x$, for some $x \in X$.

Lastly, as $\widehat{f}(h_x) = h_x(f) = f(x)$, if we identify h_x by x, then the Gelfand transform becomes the identity map $\widehat{f} = f$ on $C(X)$. ■

This result demonstrates that for a compact Hausdorff space X, the Gelfand transform on $C(X)$ is the identity map when we identify $x \in X$ by $h_x \in \sigma(C(X))$. Furthermore, the Gelfand transform preserves the norm in this case, as $\|\widehat{f}\|_{\text{sup}} = \|f\|$ for all $f \in C(X)$. This raises the question: Under what conditions does the Gelfand transform preserve the norm for any commutative unital Banach algebra?

To address this question, we first introduce the concept of an isometry, which formalizes the idea of a mapping that preserves distances.

Definition 4.18. Let $X = (X, d)$ and $\tilde{X} = (\tilde{X}, \tilde{d})$ be metric spaces. A mapping $T : X \to \tilde{X}$ is an *isometry* if $\tilde{d}(Tx_1, Tx_2) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.

With this, we can now address our previous question. Additionally, we can determine when the Gelfand transform is an isometry for a commutative unital Banach algebra.

Proposition 4.19. *Let $\mathcal{A}$ be a commutative unital Banach algebra.*

1. *If $x \in \mathcal{A}$, $\|\widehat{x}\|_{\sup} = \|x\|$ if and only if $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$.*

2. *$\Gamma_{\mathcal{A}}$ is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a commutative unital Banach algebra.

To prove the first, let $x \in \mathcal{A}$ such that $\|\widehat{x}\|_{\sup} = \|x\|$. Clearly, $\|x^{2^k}\| \leq \|x\|^{2^k}$, since the norm on a Banach algebra satisfies $\|x \cdot y\| \leq \|x\| \cdot \|y\|$. Additionally,

$$\|x\|^{2^k} = \|\widehat{x}\|_{\sup}^{2^k} = \|\widehat{x}^{2^k}\|_{\sup} = \|\widehat{x^{2^k}}\|_{\sup} \leq \|x^{2^k}\|,$$

where the first equality follows by hypothesis and the inequality follows by Theorem 4.10. Thus, $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$.

Conversely, suppose that $x \in \mathcal{A}$ and $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$. By Theorem 3.14, we have $\rho(x) = \lim_{k\to\infty} \|x^k\|^{1/k}$. Since $\|x^{2^k}\| = \|x\|^{2^k}$, it follows that

$$\rho(x) = \lim_{k\to\infty} \|x^{2^k}\|^{1/2^k} = \lim_{k\to\infty} \left(\|x\|^{2^k}\right)^{1/2^k} = \|x\|.$$

By Theorem 4.10 we have $\|\widehat{x}\|_{\sup} = \rho(x)$, and so $\|\widehat{x}\|_{\sup} = \|x\|$. Therefore, $x \in \mathcal{A}$, $\|\widehat{x}\|_{\sup} = \|x\|$ if and only if $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$.

To prove the second, suppose that $\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\sigma(\mathcal{A}))$, defined by $x \mapsto \widehat{x}$, is an isometry. Moreover, let $x \in \mathcal{A}$ be arbitrary. Then $\|x\| = \|\widehat{x}\|_{\sup}$ and so, by the previous, $\|x^2\| = \|x\|^2$, for all $x \in \mathcal{A}$.

Conversely, suppose that $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$.

For the base case, let k = 1. By assumption, $\|x^2\| = \|x\|^2$, and so $\|x^{2^1}\| = \|x\|^{2^1}$ holds for all $x \in \mathcal{A}$.

For the inductive step, suppose that $\|x^{2^k}\| = \|x\|^{2^k}$ holds for some $n = k$. Then for $n = k+1$, we see that

$$\|x^{2^{k+1}}\| = \|x^{2^k \cdot 2}\| = \|y^2\| = \|y\|^2 = \|x^{2^k}\|^2 = \|x\|^{2^k \cdot 2} = \|x\|^{2^{k+1}}.$$

Here, we substitute $x^{2^k} = y$ to yield $\|y^2\| = \|y\|^2$, which follows by assumption.

Thus, $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$ and so $\|\widehat{x}\|_{\text{sup}} = \|x\|$. Therefore, $\Gamma_{\mathcal{A}}$ is an isometry. ■

The Gelfand-Naimark theorem can be seen as an analog of Cayley's theorem for C^* algebras. Like Cayley's theorem, which says every group is isomorphic to a subgroup of S_n (the group of permutations), the Gelfand-Naimark theorem provides a concrete representation of an abstract C^* algebra.

THEOREM 4.20. *(The Gelfand-Naimark theorem). If $\mathcal{A}$ is a commutative unital C^* algebra, then the Gelfand transform $\Gamma_{\mathcal{A}}$ is an isometric $*$-isomorphism from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.*

Proof. Let $\mathcal{A}$ be a commutative unital C^* algebra. Suppose that $x \in \mathcal{A}$ and let $y = x^*x$. Then, $y^* = (x^*x)^* = x^*x^{**} = x^*x = y$, and so we have

$$\begin{aligned} \|y^{2^k}\| = \|(y^{2^{k-1}})^* y^{2^{k-1}}\| = \|y^{2^{k-1}}\|^2 = \|(y^{2^{k-2}})^* y^{2^{k-2}}\|^2 &= \|y^{2^{k-2}}\|^{2^2} \\ &= \|(y^{2^{k-3}})^* y^{2^{k-3}}\|^{2^2} \\ &\vdots \\ &= \|y\|^{2^k}. \end{aligned}$$

Thus, $\|y^{2^k}\| = \|y\|^{2^k}$ and so by Proposition 4.19 it follows that $\|\widehat{y}\|_{\text{sup}} = \|y\|$. Then,

$$\|x\|^2 = \|x^*x\| = \|y\| = \|\widehat{y}\|_{\text{sup}} = \|\widehat{x^*x}\|_{\text{sup}} = \||\widehat{x}|^2\|_{\text{sup}} = \|\widehat{x}\|_{\text{sup}}^2,$$

and so $\Gamma_{\mathcal{A}}$ is an isometry. Note that $\widehat{x^*x} = \widehat{x^*}\widehat{x} = \overline{\widehat{x}}\widehat{x} = |\widehat{x}|^2$, as $\mathcal{A}$ is symmetric by Proposition 4.14. Since $\Gamma_{\mathcal{A}}$ is an isometry, it's injective and has a closed range. By Proposition 4.14 the range of $\Gamma_{\mathcal{A}}$ is dense in $C(\sigma(\mathcal{A}))$. Since the range of $\Gamma_{\mathcal{A}}$ is closed and dense in $C(\sigma(\mathcal{A}))$, we conclude that $\text{Range}(\Gamma_{\mathcal{A}}) = C(\sigma(\mathcal{A}))$, and so $\Gamma_{\mathcal{A}}$ is surjective. Since $\Gamma_{\mathcal{A}}$ is an isometry and a bijective $*$-homomorphism, it is an isometric $*$-isomorphism from $\mathcal{A}$ to $C(\sigma(A))$. ∎

The Gelfand-Naimark theorem provides a powerful representation of commutative unital C^* algebras as algebras of continuous functions on their spectrum. As a specific example, we turn to the Gelfand transform on the Banach algebra, ℓ^1.

THEOREM 4.21. *$\sigma(\ell^1)$ can be identified with the unit circle $\mathbb{T}$ in such a way that the Gelfand transform on ℓ^1 becomes*

$$\widehat{a}(e^{i\theta}) = \sum_{-\infty}^{\infty} a_n e^{in\theta}.$$

Proof. As in Example 3.2, suppose that

$$(\delta^k)_n = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \neq k., \end{cases} \quad \text{and } \delta_n = (\delta^0)_n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Recall, by Example 3.2, multiplication in ℓ^1 is defined to be convolution. Further, $(\delta^j * \delta^k)_n = (\delta^{j+k})_n$. This follows as

$$(\delta^j * \delta^k)_n = \sum_{m=-\infty}^{\infty} \delta^j_m \delta^k_{n-m} = \sum_{m=-\infty}^{\infty} \delta_{m-j}\delta_{n-m-k},$$

where $\delta_{m-j}\delta_{n-m-k} = 1$ if and only if $j = n - k$, i.e. if $n = j + k$. Otherwise $\delta_{m-j}\delta_{n-m-k} = 0$. Therefore, $(\delta^j * \delta^k)_n = (\delta^{j+k})_n$.

Thus, ℓ^1 is generated by δ^1 and δ^{-1} and so by Proposition 4.16, $\sigma(\ell^1)$ is homeomorphic to $\sigma(\delta^1)$. We claim that $\sigma(\delta^1) = \mathbb{T}$, the multiplicative group of complex numbers of modulus 1.

To that end, consider $\lambda\delta - \delta^1$, for $\lambda \in \mathbb{C}$. If $a \in \ell^1$, then

$$
\begin{aligned}
[(\lambda\delta - \delta^1) * a]_n = (\lambda\delta * a)_n - (\delta^1 * a)_n &= \lambda(\delta * a)_n - (\delta^1 * a)_n \\
&= \lambda \sum_{m=-\infty}^{\infty} \delta_m a_{n-m} - \sum_{m=-\infty}^{\infty} \delta^1_m a_{n-m} \\
&= \lambda \sum_{m=-\infty}^{\infty} \delta_m a_{n-m} - \sum_{m=-\infty}^{\infty} \delta_{m-1} a_{n-m} \\
&= \lambda a_n - a_{n-1}.
\end{aligned}
$$

Note that the last line holds as the first sum is non-zero when $m = 0$ and the second sum in non-zero when $m = 1$. Thus, $(\lambda\delta - \delta^1) * a = \delta$ if and only if $\lambda a_0 - a_{-1} = 1$, when $n = 0$ and $\lambda a_n = a_{n-1}$ when $n \neq 0$.

Solving these equations recursively, we see that

1. $a_{-1} = \lambda a_0 - 1$,
2. $a_n = \lambda^{-1} a_{n-1} = \lambda^{-1}(\lambda^{-1} a_{n-2}) = \cdots = \lambda^{-n} a_0$, for $n \geq 0$,
3. $a_{-n} = a_{-(n)} = \lambda^{-1} a_{-(n-1)} = \lambda^{-1}(\lambda^{-1} a_{-(n-2)}) = \cdots = \lambda^{-n-1} a_{-1}$, for $n \geq 1$.

Since $\sum |a_n| < \infty$, for $n \geq 0$, we have

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |\lambda^{-n} a_0| = |a_0| \sum_{n=0}^{\infty} |\lambda^{-n}|,$$

which diverges if $|\lambda^{-1}| = |\lambda|^{-1} \geq 1$. I.e., if $|\lambda| \leq 1$. Thus a_0 must be 0.

For $n < 0$, we note that a_n is equivalent to a_{-n} with $n \geq 1$. Likewise, we have

$$\sum_{n=1}^{\infty} |a_{-n}| = \sum_{n=1}^{\infty} |\lambda^{-n-1} a_{-1}| = |\lambda^{-1} a_{-1}| \sum_{n=1}^{\infty} |\lambda^{n+1}|,$$

which will diverge if $|\lambda| \geq 1$. Thus, $a_{-1} = 0$ as $\sum |a_n| < \infty$.

Subject to these conditions, there is a unique solution if $|\lambda| \neq 1$. If $|\lambda| > 1$, we have $a = \sum_{n=0}^{\infty} \lambda^{-n-1}\delta^n$, and if $|\lambda| < 1$, we have $a = -\sum_{n=1}^{\infty} \lambda^{n-1}\delta^{-n}$.

If $|\lambda| = 1$, then the only possible solution is the trivial solution as $\sum |a_n| < \infty$ forces $a_0 = 0$ and $a_{-1} = 0$. But then, if $a_0 = 0$ the equation $a_{-1} = \lambda a_0 - 1$ implies that $a_{-1} = -1$, not zero. Thus, there is no solution when $|\lambda| = 1$.

As $\sigma(\delta^1)$ consists of $\lambda \in \mathbb{C}$ such that $\lambda\delta - \delta^1$ is not invertible, and $\lambda\delta - \delta^1$ is not invertible when $|\lambda| = 1$, it follows that $\sigma(\delta^1) = \mathbb{T}$.

By Proposition 4.16, as $\sigma(\ell^1)$ is homeomorphic to $\sigma(\delta^1)$ it follows that $\sigma(\ell^1)$ is homeomorphic to $\mathbb{T}$. Since the Gelfand transform maps elements of ℓ^1 to continuous functions on $\sigma(\ell^1)$, we can identify each $e^{i\theta} \in \mathbb{T}$ with a $h_\theta \in \sigma(\ell^1)$. Thus, for each $e^{i\theta} \in \mathbb{T}$ there exist a unique $h_\theta \in \sigma(\ell^1)$ such that $h_\theta(\delta^1) = e^{i\theta}$. Then for an element $a = \sum_{n=-\infty}^{\infty} a_n\delta^n \in \ell^1$, we have

$$h_\theta(a) = h_\theta(\sum_{n=-\infty}^{\infty} a_n\delta^n) = \sum_{n=-\infty}^{\infty} a_n h_\theta(\delta^n) = \sum_{n=-\infty}^{\infty} a_n h_\theta(\delta^1)^n = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Thus, if we identify h_θ with $e^{i\theta}$, the Gelfand transform on ℓ^1 becomes

$$\widehat{a}(e^{i\theta}) = \widehat{a}(h_\theta) = h_\theta(a) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \qquad \blacksquare$$

Corollary 4.22. *If $f(e^{i\theta}) = \sum a_n e^{in\theta}$ with $\sum |a_n| < \infty$, and f never vanishes, then $1/f(e^{i\theta}) = \sum b_n e^{in\theta}$ with $\sum |b_n| < \infty$.*

Proof. Suppose that $f(e^{i\theta}) = \sum a_n e^{in\theta}$ with $\sum |a_n| < \infty$, and f never vanishes. By assumption, $f = \widehat{a}$ with $a \in \ell^1$. As f never vanishes, a is invertible by Theorem 4.10. Let b denote its inverse. Then $\frac{1}{\widehat{a}} = \frac{1}{f} = \widehat{b}$, so $1/f(e^{i\theta}) = \widehat{b}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$. ■

Lastly, we consider the concept of a semisimple commutative unital Banach algebra.

Definition 4.23. A commutative unital Banach algebra $\mathcal{A}$ is called *semisimple* if the Gelfand transform on $\mathcal{A}$ is injective.

Example 4.24. ($\mathcal{A} = \ell^1(\mathbb{Z})$ is semisimple)

By Theorem 4.21, the Gelfand transform on $\ell^1(\mathbb{Z})$ can be identified with

$$\widehat{a}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

where $a \in \ell^1(\mathbb{Z})$. Thus, if $\widehat{a}(e^{i\theta}) = 0$ for all $\theta \in [0, 2\pi)$, then $\sum_{n=-\infty}^{\infty} a_n e^{in\theta} = 0$ for all $\theta \in [0, 2\pi)$. This implies that $a_n = 0$ for all n, and so $a = 0$ in $\ell^1(\mathbb{Z})$. Therefore, the Gelfand transform on $\ell^1(\mathbb{Z})$ is injective, and so $\ell^1(\mathbb{Z})$ is semisimple.

Another example of a semisimple commutative unital Banach algebra is $C(X)$, where X is a compact Hausdorff space. This follows by Theorem 4.17 as the Gelfand transform can be identified by the identity map, which is clearly injective.

One final remark about the algebras $C(X)$ and $\ell^1(\mathbb{Z})$ is that they are both symmetric. For $C(X)$, this follows by Theorem 4.17,

$$\widehat{f^*}(h_x) = h_x(f^*) = \overline{h_x(f)} = \bar{\widehat{f}}(h_x),$$

as the involution is given by $f^* = \bar{f}$. For $\ell^1(\mathbb{Z})$, the involution is given by $(a^*)_n = \overline{a}_{-n}$, and so

$$\widehat{a^*}(e^{i\theta}) = \sum \overline{a_{-n}} e^{in\theta} = \sum \overline{a_n} e^{-in\theta} = \overline{\widehat{a}(e^{i\theta})}.$$

4.3 Non-unital Banach algebras

In the previous section, the Gelfand transform was defined on commutative unital Banach algebras. Here, we begin to extend this work to non-unital Banach algebras.

For example, let $\mathcal{A} = L^1(\mathbb{R})$. We will show that this is a commutative Banach algebra.

Example 4.25. ($L^1(\mathbb{R})$ is a commutative Banach algebra).

It is well known that $L^1(\mathbb{R})$ is a Banach space, with respect to $\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|dx$. We now show that it is a commutative algebra under convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

First, observe that $\|f*g\|_1 \leq \|f\|_1\|g\|_1$, which follows by Theorem 2.16. Therefore, convolution is well-defined and satisfies the norm inequality.

Next, we will show convolution is commutative. Let $f, g \in L^1(\mathbb{R})$. Then,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-z)g(z)dz = (g * f)(x),$$

where we made the substitution $z = x - y$.

To show convolution is associative, let $f, g, h \in L^1(\mathbb{R})$. Then,

$$\begin{aligned}
((f * g) * h)(x) &= \int_{-\infty}^{\infty} (f * g)(y)h(x-y)dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(z)g(y-z)dz \right) h(x-y)dy \\
&= \int_{-\infty}^{\infty} f(z) \left(\int_{-\infty}^{\infty} g(y-z)h(x-y)dy \right) dz \\
&= \int_{-\infty}^{\infty} f(z) \left(\int_{-\infty}^{\infty} g(w)h(x-w-z)dw \right) dz \\
&= \int_{-\infty}^{\infty} f(z)(g * h)(x-z)dz \\
&= (f * (g * h))(x)
\end{aligned}$$

where the third equality follows from Theorem 2.14 (Fubini's theorem), and the fourth from the substitution $w = y - z$. So, we conclude that convolution is associative. Next, we will show convolution respects scalar multiplication. To that end, let $f, g \in L^1(\mathbb{R})$, and $\alpha \in \mathbb{C}$. Then,

$$(\alpha f * g)(x) = \int_{-\infty}^{\infty} \alpha f(y)g(x-y)dy = \alpha \int_{-\infty}^{\infty} f(y)g(x-y)dy = \alpha(f * g)$$

and also

$$(f * \alpha g)(x) = \int_{-\infty}^{\infty} f(y)\alpha g(x-y)dy = \alpha \int_{-\infty}^{\infty} f(y)g(x-y)dy = \alpha(f * g).$$

Thus, $\alpha f * g = f * \alpha g = \alpha(f * g)$. Lastly, for any $f, g, h \in L^1(\mathbb{R})$, we have

$$\begin{aligned}(f * (g+h))(x) &= \int_{-\infty}^{\infty} f(y)\left(g+h\right)(x-y)dy \\ &= \int_{-\infty}^{\infty} f(y)\left(g(x-y)+h(x-y)\right)dy \\ &= \int_{-\infty}^{\infty} f(y)g(x-y)dy + \int_{-\infty}^{\infty} f(y)h(x-y)dy \\ &= (f * g)(x) + (f * h)(x).\end{aligned}$$

Therefore, $L^1(\mathbb{R})$ is a commutative Banach algebra.

Moreover, $L^1(\mathbb{R})$ has no identity element under convolution. Thus, it is a non-unital algebra, and so the results from previous sections — which rely on invertibility and the spectrum — do not directly apply here. However, much of the Gelfand theory remains valid. In order to bridge this deficiency, we make use of the fact that any non-unital algebra can always be embedded in a unital algebra.

Previously, we defined a homeomorphism f between topological spaces X and Y to be a continuous bijection that has a continuous inverse. An embedding of a Banach space $\mathcal{A}$ into a Banach space $\mathcal{B}$ is a continuous, injective homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$.

Given a non-unital Banach algebra $\mathcal{A}$ we define $\tilde{\mathcal{A}}$ to be the algebra with the underlying vector space $\mathcal{A} \times \mathbb{C}$. Multiplication on $\tilde{\mathcal{A}}$ is defined by

$$(x, a) \cdot (y, b) = (xy + ay + bx, ab).$$

We can then embed $\mathcal{A}$ into $\tilde{\mathcal{A}}$ through the homomorphism $a \mapsto (a, 0)$, where $a \in \mathcal{A}$ and $(a, 0) \in \tilde{\mathcal{A}}$.

PROPOSITION 4.26. *Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\tilde{\mathcal{A}}$ is an algebra with multiplication defined by $(x, a) \cdot (y, b) = (xy + ay + bx, ab)$. Moreover, the unit element in $\tilde{\mathcal{A}}$ is* $(0, 1)$.

Proof. Suppose $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$, with multiplication defined as $(x, a) \cdot (y, b) = (xy + ay + bx, ab)$. First note, as both $\mathcal{A}$ and $\mathbb{C}$ are vector spaces, their direct product $\tilde{\mathcal{A}}$ is a vector space.

To show $\tilde{\mathcal{A}}$ is an algebra, we begin by showing multiplication is associative. To that end, consider $(x, a), (y, b), (z, c) \in \tilde{\mathcal{A}}$. Then,

$$\begin{aligned}
((x, a) \cdot (y, b)) \cdot (z, c) &= (xy + ay + bx, ab) \cdot (z, c) \\
&= ((xy + ay + bx)z + abz + c(xy + ay + bx), abc) \\
&= (x(yz + bz + cy) + bcx + a(yz + bz + cy), abc) \\
&= (x, a) \cdot (yz + bz + cy, bc) \\
&= (x, a)\,((y, b) \cdot (z, c))
\end{aligned}$$

and so multiplication is associative. Next, we will show scalar multiplication agrees with multiplication. Let $\alpha \in \mathbb{C}$ be arbitrary, and consider $(x, a), (y, b), \in \tilde{\mathcal{A}}$. Then,

$$\begin{aligned}
\alpha(x, a) \cdot (y, b) = (\alpha x, \alpha a) \cdot (y, b) = (\alpha xy + \alpha ay + \alpha bx, \alpha ab) &= \alpha(xy + ay + bx, ab) \\
&= \alpha((x, a) \cdot (y, b))
\end{aligned}$$

and similarly, $(x,a)\cdot\alpha(y,b)=\alpha((x,a)\cdot(y,b))$. Therefore,

$$\alpha(x,a)\cdot(y,b)=(x,a)\cdot\alpha(y,b)=\alpha((x,a)\cdot(y,b)).$$

Finally,

$$\begin{aligned}(x,a)\cdot((y,b)+(z,c)) &= (x,a)\cdot(y+z,b+c)\\ &= (x(y+z)+a(y+z)+(b+c)x,a(b+c))\\ &= (xy+xz+ay+az+bx+cx,ab+ac)\\ &= (xy+ay+bx,ab)+(xz+az+cx,ac)\\ &= (x,a)\cdot(y,b)+(x,a)\cdot(z,c).\end{aligned}$$

Therefore, $\tilde{\mathcal{A}}$ is an algebra. Moreover, for $(0,1)\in\tilde{\mathcal{A}}$, observe that

$$(x,a)\cdot(0,1)=(x\cdot 0+a\cdot 0+1\cdot x,a\cdot 1)=(x,a).$$

Thus, $\tilde{\mathcal{A}}$ is a unital algebra. ∎

Next, if we define the norm on $\tilde{\mathcal{A}}$ to be $\|(x,a)\|_{\tilde{\mathcal{A}}}=\|x\|_{\mathcal{A}}+|a|$ where $x\in\mathcal{A}$ and $a\in\mathbb{C}$, $\tilde{\mathcal{A}}$ becomes a Banach algebra.

PROPOSITION 4.27. *Let $\tilde{\mathcal{A}}$ be the embedding of a non-unital Banach algebra $\mathcal{A}$. Then, $\|(x,a)\|_{\tilde{\mathcal{A}}}=\|x\|_{\mathcal{A}}+|a|$ is a norm on $\tilde{\mathcal{A}}$. Moreover, this norm makes $\tilde{\mathcal{A}}$ a Banach algebra.*

Proof. Suppose that $\tilde{\mathcal{A}}$ is the embedding of the non-unital algebra $\mathcal{A}$. We first show that $\|(x,a)\|_{\tilde{\mathcal{A}}}=\|x\|_{\mathcal{A}}+|a|$ is indeed a norm on $\tilde{\mathcal{A}}$.

Clearly, $\|(x,a)\|_{\tilde{\mathcal{A}}}\geq 0$. Moreover, for any $(x,a)\in\tilde{\mathcal{A}}$, if $\|(x,a)\|_{\tilde{\mathcal{A}}}=0$ then $\|x\|_{\mathcal{A}}=0$ and $|a|=0$. Thus, $x=0$ and $a=0$ so $\|(x,a)\|_{\tilde{\mathcal{A}}}=0$ if and only if $(x,a)=0$. Therefore, $\|(x,a)\|_{\tilde{\mathcal{A}}}$ is positive definite.

Next, for any scalar λ, we have,

$$\|\lambda(x,a)\|_{\tilde{\mathcal{A}}} = \|(\lambda x, \lambda a)\|_{\tilde{\mathcal{A}}} = \|\lambda x\|_{\mathcal{A}} + |\lambda a| = |\lambda|\|x\|_{\mathcal{A}} + |\lambda||a| = |\lambda|(\|x\|_{\mathcal{A}} + |a|)$$
$$= |\lambda|\|(x,a)\|_{\tilde{\mathcal{A}}}$$

and so $\|(x,a)\|_{\tilde{\mathcal{A}}}$ satisfies absolute homogeneity. Finally, consider (x,a) and (y,a) in $\tilde{\mathcal{A}}$. Then,

$$\begin{aligned}\|(x,a)+(y,b)\|_{\tilde{\mathcal{A}}} = \|(x+y,a+b)\|_{\tilde{\mathcal{A}}} &= \|x+y\|_{\mathcal{A}} + |a+b| \\ &\leq \|x\|_{\mathcal{A}} + \|y\|_{\mathcal{A}} + |a| + |b| \\ &= (\|x\|_{\mathcal{A}} + |a|) + (\|y\|_{\dashv} + |b|) \\ &= \|(x,a)\|_{\tilde{\mathcal{A}}} + \|(y,b)\|_{\tilde{\mathcal{A}}}.\end{aligned}$$

Therefore, $\|(x,a)\|_{\tilde{\mathcal{A}}}$ also satisfies the triangle inequality, so we conclude that this is a valid norm on $\tilde{\mathcal{A}}$. Finally, as $\mathcal{A}$ is complete, it follows that $\mathcal{A} \times \mathbb{C}$ is complete and so $\tilde{\mathcal{A}}$ is a Banach algebra. ■

If we restrict the norm $\|(x,a)\|_{\tilde{\mathcal{A}}}$ to $\mathcal{A}$, we then have

$$\|(x,0)\|_{\mathcal{A}\times\{0\}} = \|x\|_{\mathcal{A}} + |0| = \|x\|_{\mathcal{A}},$$

which is the original norm on $\mathcal{A}$. Additionally, we can extend the involution on $\mathcal{A}$ to $\tilde{\mathcal{A}}$ by defining

$$(x,a)^* = (x^*, \overline{a}).$$

Moreover, $\mathcal{A} \times \{0\}$ is a closed, two-sided ideal of $\tilde{\mathcal{A}}$. This follows as for any $(x,a) \in \tilde{\mathcal{A}}$ and $(y,0) \in \mathcal{A} \times \{0\}$ we have,

$$(x,a)\cdot(y,0) = (xy + ay + 0x, a0) = (xy + ay, 0) \in \mathcal{A} \times \{0\}$$

and

$$(y, 0) \cdot (x, a) = (yx + 0x + ay, 0a) = (yx + ay, 0) \in \mathcal{A} \times \{0\}.$$

Observe that $\mathcal{A} \times \{0\}$ is closed. In the product topology, $\mathcal{A} \times \{0\}$ is the preimage of $\{0\}$ under the continuous projection map $\pi_2 : \mathcal{A} \times \mathbb{C} \to \mathbb{C}$. Since $\{0\}$ is closed in $\mathbb{C}$, its preimage $\mathcal{A} \times \{0\}$ is closed in $\tilde{\mathcal{A}}$. Since $\tilde{\mathcal{A}}/(\mathcal{A} \times \{0\}) \cong \mathbb{C}$, $\mathcal{A} \times \{0\}$ has codimension one, and so its a maximal ideal by Proposition 4.7. Thus, we identify $\mathcal{A}$ with $\mathcal{A} \times \{0\}$, viewing $\mathcal{A}$ as a maximal ideal in $\tilde{\mathcal{A}}$.

Here, we have shown how any non-unital Banach algebra $\mathcal{A}$ can be embedded into a unital Banach algebra $\tilde{\mathcal{A}}$ with a well-defined norm and multiplicative structure. This construction preserves the original properties of $\mathcal{A}$ while introducing a unit $(1, 0)$, allowing us to view $\mathcal{A}$ as a maximal ideal in $\tilde{\mathcal{A}}$. This process provides a foundation for extending tools such as the Gelfand transform, which rely on invertibility, to algebras without an identity element.

Chapter 5

Future Study

In the preceding chapter, we established the Gelfand transform on commutative Banach algebras and began extending it to non-unital algebras. This construction, embedding $\mathcal{A}$ into $\widetilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$, allows us to preserve key spectral properties while introducing a unit element.

Moving forward, we aim to study the Gelfand-Naimark theorem on non-unital C^*-algebras. Specifically, we will show that for such algebras, the Gelfand transform $\Gamma_{\mathcal{A}}$ defines an isometric $*$-isomorphism onto $C_0(\sigma(\mathcal{A}))$, the space of continuous functions vanishing at infinity on the spectrum.

We will also explore the Gelfand transform on $L^1(\mathbb{R})$. Having shown that $L^1(\mathbb{R})$ is a non-unital algebra under convolution, we will investigate how embedding into a unital algebra relates to the space $M(\mathbb{R})$ of finite Borel measures on $\mathbb{R}$, which is a Banach algebra, and how the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx$$

arises as its Gelfand transform. This would provide an interesting connection between abstract spectral theory and classical Fourier analysis.

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