Topics in Frame Theory: Frames of Translates

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Overview

Basic Definitions

A sequence $\{e_k\}_{k=1}^m$ in a vector space V is an orthonormal basis for V if:

1. $V = \text{span}\{e_k\}_{k=1}^m$,
2. $\{e_k\}_{k=1}^m$ is a linearly independent list,
3. $\langle e_k, e_j \rangle = \delta_{k,j}$,
4. For all $f \in V$, f can be uniquely represented by

$$f = \sum_{k=1}^{m} \langle f, e_k \rangle e_k.$$

Definition 1 (Frame)

A countable sequence of elements $\{f_k\}_{k \in I}$ in a separable Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B||f||^2. \quad (1)$$

- The numbers A, B are called *frame bounds*, which are not unique.
- The optimal frame bounds are achieved by taking the supremum of all lower frame bounds and the infimum of all upper frame bounds.
- We call a frame *normalized* if $||f_k|| = 1$, for all $k \in I$.

Synthesis and Analysis Operators

Synthesis Operator

Define a linear mapping

$$T : \mathbb{C}^m \to V,\ T(\{c_k\}_{k=1}^m) = \sum_{k=1}^{m} c_k f_k.$$

This operator T is called the *synthesis operator*, or the *pre-frame operator*.

Analysis Operator

The adjoint operator of T is given by

$$T^* : V \to \mathbb{C}^m, \; T^* f = \{\langle f, f_k \rangle\}_{k=1}^m,$$

is called the *analysis operator*.

Frame Operator

When we compose T with its adjoint T^* we obtain the *frame operator*

$$S : V \to V,\ Sf = TT^*f = \sum_{k=1}^{m} \langle f, f_k \rangle f_k.$$

This also means that, in terms of the frame operator,

$$\langle Sf, f \rangle = \sum_{k=1}^{m} |\langle f, f_k \rangle|^2, f \in V,$$

Definition 2

A frame $\{f_k\}_{k=1}^{m}$ in a Hilbert space $\mathcal{H}$ of dimension n is overcomplete if $m > n$.

If our frame is overcomplete, it is possible that it remains a frame after the deletion of an element.

Natural Question

How do we construct frames that are stable toward the removal of more than one element?

Simple Example

Let $\{e_1, e_2\}$ be an orthonormal basis for $\mathbb{C}^2$ The sequence $\{e_1, e_2, e_1 + e_2, e_1 - e_2\}$ is a frame for $\mathbb{C}^2$ with *excess* 2. This means that one can delete 2 *arbitrary* vectors and still be a frame for $\mathbb{C}^2$.

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Example

Harmonic Frame

Let $m > n$ and define the vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ by:

$$f_k = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 \\ e^{2\pi i \frac{k-1}{m}} \\ ... \\ e^{2\pi i (n-1) \frac{k-1}{m}} \\ ... \\ e^{2\pi i (m-1) \frac{k-1}{m}} \end{pmatrix}, k = 1, ..., m$$

This is an overcomplete Parseval frame for $\mathbb{C}^n$ and $||f_k|| = \sqrt{\frac{n}{m}}$ for all $k = 1, ..., m$. This frame also has the nice property that you can delete $m - n$ elements from it and it will still be a frame for $\mathbb{C}^n$.

Mercedes-Benz

Construction

One of the non-trivial examples of a frame in $\mathbb{R}^2$ is the Mercedes-Benz frame. This frame $\{f_k\}_{k=1}^3$ is given by the vectors

$$f_1 = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix},\ f_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}, f_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

The synthesis operator is given by the matrix with the frame vectors as the columns:

$$T = \begin{pmatrix} | & | & | \\ f_1 & f_2 & f_3 \\ | & | & | \end{pmatrix}$$

Then the frame operator $S = TT^*$ is given by the matrix multiplication:

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \sqrt{\frac{2}{3}} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means that any vector $f \in \mathbb{R}^2$ can be written as

$$f = Sf = TT^* f = \sum_{k=1}^{3} \langle f, f_k \rangle f_k.$$

Infinite Dimensional Setting

Definition 3 (Translation Operator)

For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (T_a f)(x) := f(x-a), x \in \mathbb{R}.$$

The translation operator satisfies the following conditions:

1. $T_a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is unitary for all $a \in \mathbb{R}$, and
2. For each $f \in L^2(\mathbb{R})$, the mapping $a \longmapsto T_a f$ is continuous from $\mathbb{R}$ to $L^2(\mathbb{R})$.

Definition 4 (Frame of Translates)

A system of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ which is a frame for $\mathcal{H}$ is called a frame of translates where T_{λ_k} is the translation by $\lambda_k \in \mathbb{R}$ and ϕ is a function from $L^2(\mathbb{R})$.

Frame of Translates Example

We often think of the exponential function

$$e_n(x) = e^{2\pi inx}$$

as being a 1-periodic function on $\mathbb{R}$. This function does not belong to $L^2(\mathbb{R})$, however, since $|e^{2\pi inx}| = 1$ for every x.

Frame of Translates Example

Consider the functions restricted to $[-\frac{1}{2}, \frac{1}{2}]$

$$\epsilon_n = e_n \cdot \chi_{[-\frac{1}{2}, \frac{1}{2}]} \in L^2(\mathbb{R}).$$

The interval $[-\frac{1}{2}, \frac{1}{2}]$, which we will denote as I, has length 1 and $\{\epsilon_n\}_{n \in \mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$, we take the closed span of this sequence to get

$$L_I^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : f(x) = 0 \text{ for a.e. } |x| > \frac{1}{2} \right\}.$$

Frames of Translates Example

Letting

$$\phi(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$$

we can write the Fourier transform of ϵ_n as a translation of $\frac{\sin(x)}{x}$

$$\widehat{\epsilon_n}(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i n x} e^{-2\pi i \xi x} dx = \frac{\sin(\pi(\xi - n))}{\pi(\xi - n)} = T_n\phi(\xi).$$

Payley-Wiener Spaces

The functions $T_n\phi$ are not compactly supported but each has a Fourier transform which is nonzero only within $[-\frac{1}{2}, \frac{1}{2}]$. We take the closed span of the orthonormal sequence $\{\hat{\epsilon}_n\}_{n\in\mathbb{Z}} = \{T_n\phi\}_{n\in\mathbb{Z}}$ to be

Definition 5 (Payley-Wiener Space)

$$PW(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R})) : \hat{f}(\xi) = 0 \text{ for a.e. } |\xi| > \frac{1}{2} \right\}.$$

Main Result

Theorem 6

A system of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ *is never a frame for* $L^2(\mathbb{R})$*, regardless of the choice of the function* $\phi \in L^2(\mathbb{R})$ *and sequence* $\{\lambda_k\}_{k\in\mathbb{Z}}$.

Definition 7 (Bessel Sequence)

A sequence $\{f_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B||f||^2$$

Definition 8 (Beurling Density)

Given a sequence $\{\lambda_k\}_{k\in I} \in \mathbb{R}^d$, for $x \in \mathbb{R}$ and $h > 0$, we let

$$Q_h(x) = \prod_{j=1}^{d} \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right), \text{ where } x = (x_1, ..., x_d),$$

denote the half-open cube in $\mathbb{R}^d$ centered at x with side lengths h.

Beurling Density

Denote the largest number of points from $\{\lambda_k\}_{k\in I}$ that lie an any cube $Q_h(x)$ by $v^+(h) = \sup_{x\in\mathbb{R}^d} \natural(\{\lambda_k\}_{k\in I} \cap Q_h(x))$, where $\natural$ denotes the cardinality of a set. The upper Beurling density of $\{\lambda_k\}_{k\in I}$ are defined as

$$D^+(\{\lambda_k\}_{k\in I}) = \limsup_{h\to\infty} \frac{v^+(h)}{h^d}.$$

Relatively Separated and Separated Sequences

Let I be a countable index set and $\{\lambda_k\}_{k\in I} \in \mathbb{R}^d$. We say that

1. a point $\lambda \in \mathbb{R}^d$ is an accumulation point for $\{\lambda_k\}_{k\in I}$ if every open ball in $\mathbb{R}^d$ centered at λ contains infinitely many λ_k,
2. $\{\lambda_k\}_{k\in I}$ is separated if $\inf_{j\neq k} |\lambda_j - \lambda_k| > 0$ and there exists a constant $\delta > 0$ such that $|\lambda_j - \lambda_k| \geq \delta$ for all $j \neq k$ is called a separation constant, and
3. $\{\lambda_k\}_{k\in I}$ is relatively separated if it is a finite union of separated sequences.

Important Lemma

Lemma 9

Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a sequence in* $\mathbb{R}^d$*. Then the following are equivalent:*

1. $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$
2. *For some* $h > 0$ *there is a natural number* N_h *such that each cube* $Q_h(hn)$, $n \in \mathbb{Z}^d$, *contains at most* N_h *points from* $\{\lambda_k\}_{k\in\mathbb{Z}}$, *i.e.,*

$$\sup_{n\in\mathbb{Z}^d} \natural(\{\lambda_k\}_{k\in\mathbb{Z}} \cap Q_h(hn)) < \infty.$$

3. $\{\lambda_k\}_{k\in\mathbb{Z}}$ *is relatively separated [1].*

Key Idea

$\{\lambda_k\}_{k\in\mathbb{Z}}$ can be split into h-separated sequences. Let $e_1, ..., e_{2^d}$ denote the vertices of the unit cube $[0,1]^d$, and consider the sets

$$Z_j = (2\mathbb{Z})^d + e_j$$

where $j = 1, ..., 2^d$. Then $\mathbb{Z}^d$ is the disjoint union of the sets $Z_1, ..., Z_{2^d}$. Since $\{Q_h(hn)\}_{n\in\mathbb{Z}^d}$ is a disjoint cover of $\mathbb{R}^d$, this means that $\mathbb{R}^d$ is the disjoint union of the sets

$$B_j = \bigcup_{n\in Z_j} Q_h(hn), j = 1, ..., 2^d.$$

Proving Main Result

Theorem 10

A system of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is never a frame for $L^2(\mathbb{R})$, regardless of the choice of the function $\phi \in L^2(\mathbb{R})$ and sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ [1].

1. If $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) = \infty$, then $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is not a Bessel sequence.
2. If $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$, then $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ then the lower frame condition fails.

Important Note

- $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ must be a Bessel sequence to be a frame.
- If $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$, then $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is relatively separated.

Highlights of Proof

Choose a separation constant $\delta > 0$ for each sequence $\{\lambda_k\}_{k \in I_j}$, $j = 1, ..., s$, and consider $h \in \left(0, \frac{\delta}{2}\right)$. With $I := [-h, h]$,

$$\sum_{k \in \mathbb{Z}} |\langle \chi_I, T_{\lambda_k} \phi \rangle|^2 \leq ||\chi_I||^2 \sum_{j=1}^{s} \int_{\Delta_j} |\phi(x)|^2 dx.$$

Applying Lebesgue's dominated convergence theorem shows that for each fixed $j = 1, ..., s$,[1]

$$\int_{\Delta_j} |\phi(x)|^2 dx \to 0 \text{ as } h \to 0.$$

This implies that the lower frame bound $A = 0$.

[1] Where $\Delta_j = \bigcup_{k \in I_j} (I - \lambda_k)$

- If the Beurling density is infinite, then our sequence $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is not a Bessel sequence.
- If the Beurling density is finite, our sequence $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ violates the lower frame condition.
- We conclude $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ cannot be a frame for $L^2(\mathbb{R})$.

References

[1] Ole Christensen, *An introduction to frames and Riesz bases. Applied and Numerical Harmonic Analysis.* Birkhäuser Boston, Inc., Boston, MA, 2003.

Questions

$$\begin{aligned} A||\chi_I||^2 =& A(4h^2) \\ \leq& \sum_{k\in\mathbb{Z}} |\langle \chi_I, T_{\lambda_k}\rangle|^2 \\ \leq& 4h^2 \sum_{j=1}^{s} \int_{\Delta_j} |\phi(x)|^2 dx \end{aligned}$$

$$A||\chi_I||^2 \leq ||\chi_I||^2 s \cdot \max_j \int_{\Delta_j} |\phi(x)|^2 dx \to 0.$$