Topics in Frame Theory: Frames of Translates

by

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An Alternate Plan Paper Submitted in Partial Fulfillment of the Requirements for

Masters of Arts

In Mathematics

Minnesota State University, Mankato

Mankato, Minnesota

May, 2024

ACKNOWLEDGEMENTS

Firstly, I would like to thank my academic advisor, Kristopher Hollingsworth. I appreciate your patience and help throughout the entire process of completing my APP. Thank you for the excellent explanations and for helping me achieve a deeper understanding of my research area. I would also like to thank the faculty members in the math department who were always there to answer questions about anything I was working on or wanted to know about.

To my academic cohort, thank you for being amazing friends throughout my two years in the program. To Jack, Veronika, Oksana, Lucas, Bobby, and Brenden - thank you for all of the fun times we had talking in the MLC about anything that came to mind. Thank you Rebecca for studying with me all summer for the comprehensive exam. Thank you to Lizzy, Mingyu, Xeng, and Tati for inviting everyone to play volleyball every weekend, it was a fun time. Also thank you to Jimmy for convincing everyone to play Magic, it was an awesome experience to have with you, Xeng, Brenden, Christian, and Jack.

To my family, thank you for your support during my time in the master's program. Sending me funny videos every now and then kept me motivated throughout my stay in Mankato. I look forward to being in Bismarck with all of you again.

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Abstract

We examine the construction of frames in a finite-dimensional Hilbert space $\mathcal{H}$. Specific examples include the Mercedes-Benz frame for $\mathbb{R}^2$ and the harmonic frame for $\mathbb{C}^n$. In infinite-dimensional settings, we take a look at frames of translates and their application. We also present a proof as to why other types of frame constructions are necessary for the function space $L^2(\mathbb{R})$.

Chapter 1

Introduction

While studying vector spaces, one of the basic concepts introduced is a basis. This provides a way of writing an expansion of every vector in the space as a linear combination of basis vectors. The concept of basis is important but the conditions required are restrictive since you must find a list of linearly independent vectors. This means that wanting a basis to satisfy additional conditions is not always possible.

In Chapter 2, we will discuss the concept and construction of frames in finite-dimensional Hilbert spaces. Here we will see some of the notation that will be used throughout the paper and also a general overview of frames and their properties. After these properties are discussed we will look at specific frame constructions in different finite-dimensional Hilbert spaces.

In Chapter 3 we will discuss frames in the infinite-dimensional setting. Here we discuss frames of translates and their applications in signal processing. Then we conclude this section by showing that frames of translates are useful but cannot produce a frame for $L^2(\mathbb{R})$.

In the final Chapter, we explore some avenues through which one could continue researching frame theory. Here we give different examples and papers referring to a variety of topics that can be researched in the field of frame theory.

Chapter 2

Technical Background and Finite Dimensional Frames

For finite dimensions, one of the major concepts that we will compare frames with will be an orthonormal basis. A sequence $\{e_k\}_{k=1}^m$ in a vector space V is an orthonormal basis for V if

(i) $V = \text{span}\{e_k\}_{k=1}^m$,

(ii) $\{e_k\}_{k=1}^m$ is a linearly independent list,

(iii) $\langle e_k, e_j \rangle = \delta_{k,j}$ where $\delta_{k,j}$ is the Kroneker-delta function, and

(iv) for all $f \in V$, f can be uniquely represented by

$$f = \sum_{k=1}^{m} \langle f, e_k \rangle e_k.$$

For our purpose, we want to be able to have the same sort of reconstruction that an orthonormal basis gives us. We also want to lift some of the restrictions that an orthonormal basis has, such as being a minimal set and forcing orthogonality. Later we will see that by adding redundancy to our spanning set, we can achieve some desired properties while still preserving a method of stable reconstruction. For now, we will introduce the concept of a frame.

Definition 2.1 (Frame). A countable sequence of elements $\{f_k\}_{k \in I}$ in V is a frame for V if there exist positive real numbers A, B such that for all $f \in V$

$$A||f||^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B||f||^2.$$

The numbers A, B are called lower and upper frame bounds, respectively, which are not unique. The optimal frame bounds are achieved by taking the supremum of all lower frame bounds and the infimum of all upper frame bounds. Lastly, we call a frame *normalized* if $||f_k|| = 1$, for all $k \in I$.

In finite dimensions, the upper frame condition is automatically satisfied for any finite set of vectors, by the Cauchy-Schwarz inequality

$$\sum_{k=1}^{m} |\langle f, f_k\rangle|^2 \leq \sum_{k=1}^{m} ||f_k||^2||f||^2,$$

but the upper bound given by this is usually not tight.

To ensure the lower frame bound condition is satisfied, it is necessary that our set of vectors $\{f_k\}_{k=1}^m$ span V, i.e., span$\{f_k\}_{k=1}^m = V$.

LEMMA 2.2. *Let $\{f_k\}_{k=1}^m$ be a sequence in a finite-dimensional vector space V. Then $\{f_k\}_{k=1}^m$ is a frame for the vector space W :=span$\{f_k\}_{k=1}^m$.*

Proof. Assuming that not all f_k are zero, we know the upper frame condition is satisfied by the Cauchy-Schwarz inequality with $B = \sum_{k=1}^m ||f_k||^2||f||^2$. To show that the lower frame bound $A > 0$ exists, consider the continuous mapping:

$$\phi : W \to \mathbb{R},\ \phi(f) := \sum_{k=1}^{m} |\langle f, f_k\rangle|^2.$$

Since W is finite dimensional, the unit sphere in W is compact, so we can find $g \in W$ where $||g|| = 1$ such that

$$A := \sum_{k=1}^{m} |\langle f, f_k\rangle|^2 = \inf\Big\{\sum_{k=1}^{m} |\langle f, f_k\rangle|^2 : f \in W, ||f|| = 1\Big\}.$$

Now we have $A > 0$ and given $f \in W$, $f \neq 0$ it follows that

$$\sum_{k=1}^{m} |\langle f, f_k\rangle|^2 = \sum_{k=1}^{m} \Big|\Big\langle \frac{f}{||f||}, f_k\Big\rangle\Big|^2 ||f||^2 \geq A||f||^2.$$

∎

The immediate consequence of this is that a sequence of vectors $\{f_k\}_{k=1}^m$ in a finite-dimensional vector space V is a frame for V if and only if $\operatorname{span}\{f_k\}_{k=1}^m = V$. This also means that every basis in a finite-dimensional vector space is also a frame for the space. We call a frame which is not a basis for the space overcomplete or redundant.

Now we will consider a vector space V with a frame $\{f_k\}_{k=1}^m$ and define a linear mapping

$$T : \mathbb{C}^m \to V,\ T(\{c_k\}_{k=1}^m) = \sum_{k=1}^m c_k f_k.$$

This operator T is called the *synthesis operator*, or the *pre-frame operator*. The adjoint operator of T is given by

$$T^* : V \to \mathbb{C}^m,\ T^* f = \{\langle f, f_k\rangle\}_{k=1}^m,$$

is called the *analysis operator*. When we compose T with its adjoint T^* we obtain the *frame operator*

$$S : V \to V,\ Sf = TT^* f = \sum_{k=1}^m \langle f, f_k\rangle f_k.$$

This also means that, in terms of the frame operator, for any $f \in V$ that

$$\langle Sf, f\rangle = \sum_{k=1}^m |\langle f, f_k\rangle|^2.$$

Moving on to some properties of the frame operator we have the following theorem.

THEOREM 2.3 (Theorem 1.1.5 in [1]). *Let $\{f_k\}_{k=1}^m$ be a frame for a finite-dimensional vector space V with frame operator S. Then the following hold:*

(i) S is invertible and self-adjoint.

(ii) Every $f \in V$ can be represented as

$$f = \sum_{k=1}^m \langle f, S^{-1} f_k\rangle f_k = \sum_{k=1}^m \langle f, f_k\rangle S^{-1} f_k.$$

(iii) If $f \in V$ also has the representation $f = \sum_{k=1}^{m} c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^{m}$, then

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} |\langle f, S^{-1} f_k \rangle|^2 + \sum_{k=1}^{m} |c_k - \langle f, S^{-1} f_k \rangle|^2.$$

Proof. Since $S = TT^*$, S is self adjoint. To prove S is injective, let $f \in V$ and assume that $Sf = 0$. Then

$$0 = \langle Sf, f \rangle = \sum_{k=1}^{m} |\langle f, f_k \rangle|^2.$$

This would imply that f must be 0 since $\{f_k\}_{k=1}^{m}$ is a frame for V it must have a lower frame bound $A > 0$. To show S is surjective, note the existence of the lower frame bound $A > 0$. This means that the frame $\{f_k\}_{k=1}^{m}$ spans V, meaning the synthesis operator T is surjective. Given $f \in V$ we can find $g \in \mathbb{C}^m$ such that $Tg = f$. We can choose $g \in \text{null}(T)^{\perp} = \text{range}(T^*)$, it follows that the $\text{range}(S) = \text{range}(TT^*) = V$, thus S is surjective.

To obtain the representation from (ii), each $f \in V$ can be represented by

$$f = SS^{-1} f = \sum_{k=1}^{m} \langle S^{-1} f, f_k \rangle f_k.$$

Using the fact that S is self adjoint, we can write

$$f = S^{-1} S f = \sum_{k=1}^{m} \langle f, S^{-1} f_k \rangle f_k$$

as desired.

To prove statement (iii), suppose that $f = \sum_{k=1}^{m} c_k f_k$. Then we can write

$$\{c_k\}_{k=1}^{m} = (\{c_k\}_{k=1}^{m} - \{\langle f, S^{-1} f_k \rangle\}_{k=1}^{m}) + \{\langle f, S^{-1} f_k \rangle\}_{k=1}^{m}.$$

By the choice of $\{c_k\}_{k=1}^{m}$, we have

$$\sum_{k=1}^{m} (c_k - \langle f, S^{-1} f_k \rangle) f_k = 0,$$

meaning $\{c_k\}_{k=1}^{m} = (\{c_k\}_{k=1}^{m} - \{\langle f, S^{-1} f_k \rangle\}_{k=1}^{m}) \in \text{null}(T)^{\perp} = \text{range}(T^*)$. Since our sequence of coefficients $\{\langle f, S^{-1} f_k \rangle\}_{k=1}^{m} = \{\langle S^{-1} f, f_k \rangle\}_{k=1}^{m} \in \text{range}(T^*)$, proving statement (iii) ∎

Since every frame in a finite-dimensional vector space is a spanning set, if the frame is redundant there are many ways to represent a single vector $f \in V$. More pointedly, if $\{f_k\}_{k=1}^m$ is a redundant frame for V, there exist nonzero sequences $\{d_k\}_{k=1}^m$ such that $\sum_{k=1}^m d_k f_k = 0$. This means we can write any element $f \in V$ as

$$\begin{aligned} f &= \sum_{k=1}^{m} \langle f, S^{-1} f_k \rangle f_k + \sum_{k=1}^{m} d_k f_k \\ &= \sum_{k=1}^{m} (\langle f, S^{-1} f_k \rangle + d_k) f_k. \end{aligned}$$

This shows that given a redundant frame, any $f \in V$ has many different representations. The coefficients $\{\langle f, S^{-1} f_k \rangle\}_{k=1}^m$ are often chosen since they have minimal ℓ^2-norm among all of the sequences $\{c_k\}_{k=1}^m$ for which $f = \sum_{k=1}^m c_k f_k$. The numbers $\langle f, S^{-1} f_k \rangle$, $k = 1, ..., m$ are called *frame coefficients*. Also note that since the frame operator S is bijective, $\{S^{-1} f_k\}_{k=1}^m$ is also a frame generally called the canonical dual frame of $\{f_k\}_{k=1}^m$.

One of the non-trivial examples of a frame in $\mathbb{R}^2$ is the Mercedes-Benz frame. This frame $\{f_k\}_{k=1}^3$ is given by the vectors

$$f_1 = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}, \qquad f_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}, \qquad f_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

The synthesis operator is given by the matrix with the frame vectors as columns

$$\begin{pmatrix} | & | & | \\ f_1 & f_2 & f_3 \\ | & | & | \end{pmatrix}$$

Then the synthesis operator $S = TT^*$ is given by the matrix multiplication

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \sqrt{\frac{2}{3}} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means that any vector $f \in \mathbb{R}^2$ can be written as

$$f = Sf = TT^*f = \sum_{k=1}^{3} \langle f, f_k \rangle f_k.$$

These vectors are scaled to yield an identity operator S, but this will lead us to a more general characterization of frames with the following theorem.

THEOREM 2.4. *Let $\{f_k\}_{k=1}^m$ be a frame for a d-dimensional vector space V with frame operator S. Then the following hold.*

(i) The optimal lower frame bound is the smallest eigenvalue of S, and the optimal upper frame bound is the largest eigenvalue of S.

(ii) If $\{f_k\}_{k=1}^m$ is a tight frame and for all k such that $||f_k|| = 1$, then the frame bound is $A = \dfrac{m}{d}$.

Proof. Since the frame operator $S : V \to V$ is self-adjoint, there is some orthonormal basis of V consisting of eigenvectors of S. Let the basis of eigenvectors be denoted by $\{e_i\}_{i=1}^d$ with corresponding eigenvalues $\{\lambda_i\}_{i=1}^d$. Then every vector $f \in V$ can be written as

$$f = \sum_{i=1}^{d} \langle f, e_i \rangle e_i.$$

Then

$$Sf = \sum_{i=1}^{d} \langle f, e_i \rangle Se_i = \sum_{i=1}^{d} \lambda_i \langle f, e_i \rangle e_i,$$

and

$$\sum_{k=1}^{m} |\langle f, f_k \rangle|^2 = \langle Sf, f \rangle = \sum_{i=1}^{d} \lambda_i |\langle f, e_i \rangle|^2.$$

Therefore

$$\lambda_{\min} ||f||^2 \le \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \le \lambda_{\max} ||f||^2.$$

This shows that $\lambda_{\min}$ and $\lambda_{\max}$ are lower and upper frame bounds. Choosing f to be an eigenvector corresponding to $\lambda_{\min}$ and $\lambda_{\max}$ respectively proves that these values are the optimal frame bounds for our frame, which proves our first statement (i).

Moving on to statement (ii) first observe that

$$\sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \lambda_i ||e_i||^2 = \sum_{i=1}^{d} \langle Se_i, e_i \rangle = \sum_{i=1}^{d} \sum_{k=1}^{m} |\langle e_i, f_k \rangle|^2 = \sum_{k=1}^{m} ||f_k||^2.$$

Assuming that $\{f_k\}_{k=1}^{m}$ is a tight frame, $S = AI$, meaning S only has a single eigenvalue which is equal to A which has a multiplicity of d. This means that

$$dA = m$$

which gives us that $A = \dfrac{m}{d}$. ■

A nice observation and consequence of this is that given our Mercedes-Benz frame construction, we can normalize each f_k and arrive at an optimal frame bound of $\frac{3}{2}$. In frame theory, these frame bounds help determine how stable your reconstructions would be. By studying the ratio $\frac{B}{A}$, called the condition number, we can gather some understanding of how fast various numerical algorithms will take to converge if calculating the frame coefficients is computationally inefficient. This motivates us to construct frames with a *condition number* as close to 1 as possible, where it is identically 1 when the frame is tight.

Application of frames often asks for the frames to be tight as the inversion of the frame operator is oftentimes extremely inefficient, and tight frame constructions avoid inverting the frame operator during reconstruction. We will now look at a different type of frame construction called the harmonic frame.

Example 2.5. Let $m > n$ and define the vectors $\{f_k\}_{k=1}^m \in \mathbb{C}^m$ by

$$f_k = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 \\ e^{2\pi i \frac{k-1}{m}} \\ \vdots \\ \vdots \\ e^{2\pi i (n-1)\frac{k-1}{m}} \end{pmatrix}.$$

The sequence $\{f_k\}_{k=1}^m$ is an overcomplete 1-tight frame (called a *Parseval frame*) for $\mathbb{C}^n$ where $||f_k|| = \sqrt{\frac{n}{m}}$ for all $k = 1..., m$. This specific frame construction is useful in signal processing as we will show it is stable under the deletion of elements. The harmonic frame has the property that any set of n vectors from $\{f_k\}_{k=1}^m$ is a basis for $\mathbb{C}^n$. This property is called full spark, or maximal robustness.

Proposition 2.6. *Consider the harmonic frame $\{f_k\}_{k=1}^m \in \mathbb{C}^n$. Any subset containing at least n elements of this frame forms a frame for $\mathbb{C}^n$.*

Proof. Consider an arbitrary subset $\{k_1, k_2, ..., k_n\} \subseteq \{1, 2, ..., m\}$. Placing the vectors $\{f_{k_i}\}_{i=1}^n$ as rows in an $n \times n$ matrix and letting $z := e^{\frac{2\pi i}{m}}$ we obtain

$$\begin{pmatrix} -f_{k_1}- \\ -f_{k_2}- \\ \vdots \\ \vdots \\ -f_{k_n}- \end{pmatrix} = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & z^{k_1-1} & \dots & z^{(k_1-1)(n-1)} \\ 1 & z^{k_2-1} & \dots & z^{(k_2-1)(n-1)} \\ \vdots & \dots & \ddots & \vdots \\ 1 & z^{k_n-1} & \dots & z^{(k_n-1)(n-1)} \end{pmatrix}$$

This is a Vandermonde matrix with determinant

$$\frac{1}{m^{\frac{n}{2}}} \prod_{i,j=1, i\neq j}^{n} (z^{k_i-1} - z^{k_j-1}) \neq 0,$$

Where proof of this fact can be found in proposition 1.5.7 in [1]. Thus $\{f_{k_i}\}_{i=1}^n$ is a basis for $\mathbb{C}^n$. ■

In signal processing, there can be many different real-world scenarios in which packets of information can be lost during transmission. Under this assumption, if we view the frame vectors to be these packets of information and some of these packets are lost, a construction similar to the harmonic frame is optimal. Being maximally robust or full spark is a valuable property to have in these settings as the construction can lose any set of $n-m$ elements and still be a basis for the space, which would not be true for every valid frame construction on $\mathbb{C}^n$.

Chapter 3

Frames in Infinite Dimensional Space

To generalize the concept of frames we must move from the finite-dimensional Hilbert spaces to infinite-dimensional ones. For this paper we restrict our attention to

$$L^2(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} |f|^2 < \infty \right\}.$$

There are three main types of frame constructions in infinite dimensions. These are

1. Frames of Translates,
2. Gabor Frames, and
3. Wavelet Frames.

Each frame construction is done by applying different types of operators to a single function in $L^2(\mathbb{R})$. Frames of Translates are constructed by applying a translation operator to a function and this is the only construction that uses a single type of operator for their representations. Gabor Frames use translations and also introduce a modulation operator in their construction and Wavelet Frames use translations and introduce a dilation operator as well. Our focus will be on Frames of Translates as their construction and subsequent limitations forced researchers to discover alternate ways to construct stable frames in $L^2(\mathbb{R})$.

To discuss Frames of Translates, we will define the translation operator:

Definition 3.1 (Translation Operator). For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (T_a f)(x) := f(x-a).$$

Let us look at some characteristics of the translation operator with the following lemma.

Lemma 3.2 (Lemma 2.9.2 in [1]). *The translation operator satisfies the following conditions:*

(i) $T_a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ *is unitary for all* $a \in \mathbb{R}$

(ii) For each $f \in L^2(\mathbb{R})$*, the mapping* $a \longmapsto T_a f$ *is continuous from* $\mathbb{R}$ *to* $L^2(\mathbb{R})$*.*

Proof. To show that T_a is a unitary operator,

$$\langle T_a f, g \rangle = \int_{-\infty}^{\infty} f(x-a)\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)\overline{g(x+a)}dx = \langle f, T_{-a} g \rangle,$$

for all f, $g \in L^2(R)$. Clearly $T_a^{-1} = T_{-a}$ since

$$T_a T_{-a} f(x) = T_a(f(x+a)) = f(x+a-a) = f(x)$$

thus T_a is invertible, which proves (i).

To prove (ii), suppose that f is a continuous function with compact support in the bounded interval $[a,b]$. Since f is uniformly continuous, given $\epsilon > 0$ choose $\delta > 0$ such that

$$|f(x-y) - f(x)| \leq \epsilon \text{ for all } x \in \mathbb{R} \text{ whenever } |y| \leq \delta.$$

For this chosen δ, we see that for $c \leq d \in [a,b]$,

$$||T_a f - T_{a_0} f|| = \left(\int_{-\frac{1}{2}+c}^{\frac{1}{2}+d} |f(x-y) - f(x)|^2 dx \right)^{\frac{1}{2}} \leq \epsilon\sqrt{d-c+1}.$$

Thus $a \mapsto T_a f$ is continuous for a continuous function f with compact support. To generalize this argument to an arbitrary function $f \in L^2(\mathbb{R})$, we use the fact that continuous functions with compact support are dense in $L^2(\mathbb{R})$. We can approximate f arbitrarily close with continuous functions that have compact support which implies that f is continuous concluding the proof of (ii). ∎

The translation operator behaves well and is beneficial when attempting to have a stable frame construction. A system of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is called a frame of translates where T_{λ_k} is the translation by λ_k and ϕ is a function from $L^2(\mathbb{R})$. Frames of this form are studied in signal processing and we will now look at an example of this.

We often think of the exponential function

$$e_n(x) = e^{2\pi inx}$$

as being a 1-periodic function on $\mathbb{R}$ This function does not belong to $L^2(\mathbb{R})$, however, since $|e^{2\pi inx}| = 1$ for every x. What we can do to remedy this is restrict our domain to be of length 1 and symmetric about the origin, namely we are considering the functions

$$\epsilon_n = e_n \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]} \in L^2(\mathbb{R}).$$

Where $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ is the characteristic function. The interval $[-\frac{1}{2},\frac{1}{2}]$, which we will denote as I has length 1 and $\{\epsilon_n\}_{n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$, taking the closed span of this sequence to get

$$L^2_I(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) : f(x) = 0 \text{ for a.e. } |x| > \frac{1}{2}\right\}.$$

We refer to $L^2_I(\mathbb{R})$ as the subspace of $L^2(\mathbb{R})$ consisting of functions that are "timelimited" to the interval I. Since the Fourier transform is unitary we can also discuss the sequence $\{\widehat{\epsilon_n}\}_{n\in\mathbb{Z}}$ which is orthonormal in $L^2(\mathbb{R})$. Letting

$$\phi(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$$

we can write the Fourier transform of ϵ_n as a translated $\frac{\sin(x)}{x}$ function:

$$\hat{\epsilon_n}(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi inx}e^{-2\pi i\xi x}dx = \frac{\sin(\pi(\xi - n))}{\pi(\xi - n)} = T_n\phi(\xi).$$

The functions $T_n\phi$ are not compactly supported but each has a Fourier transform which is nonzero only within $[-\frac{1}{2},\frac{1}{2}]$. We take the closed span of the orthonormal sequence

$\{\hat{\epsilon}_n\}_{n\in\mathbb{Z}} = \{T_n\phi\}_{n\in\mathbb{Z}}$ to be

$$PW(\mathbb{R}) = \Big\{f \in L^2(\mathbb{R})) : \hat{f}(\xi) = 0 \text{ for a.e. } |\xi| > \frac{1}{2}\Big\}.$$

We call $PW(\mathbb{R})$ the *Payley-Wiener space* of functions "bandlimited" to the interval $[-\frac{1}{2}, \frac{1}{2}]$. This space is used widely in signal processing and sampling theory and is created by taking the closed span of translations of functions which gives a nice example of frames of translates. Now we will move into the major limiting factor of these types of frames which causes them to be insufficient in $L^2(\mathbb{R})$, namely that there is no frame of translates for $L^2(\mathbb{R})$. To reach this result we must introduce a few more concepts which will be needed for the proof. We will now introduce a few definitions and lemmas.

Definition 3.3 (Bessel Sequence). A sequence $\{f_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B||f||^2.$$

Bessel sequences are frames if they also satisfy the lower frame condition, as any Bessel sequence automatically satisfies the upper frame bound condition. Secondly, we will discuss the notion of Beurling density.

Definition 3.4 (Beurling Density). Given a sequence $\{\lambda_k\}_{k\in I} \in \mathbb{R}^d$, for $x \in \mathbb{R}^d$ and $h > 0$ where $x = (x_1, \ldots, x_d)$, we let

$$Q_h(x) = \prod_{j=1}^{d} \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right), ,$$

denote the half-open cube in $\mathbb{R}^d$ centered at x with side lengths h. Denote the largest and smallest number of points from $\{\lambda_k\}_{k\in I}$ that lie an any cube $Q_h(x)$ by

$$v^+(h) = \sup_{x\in\mathbb{R}^d} \natural(\{\lambda_k\}_{k\in I} \cap Q_h(x))$$

and

$$v^-(h) = \inf_{x\in\mathbb{R}^d} \natural(\{\lambda_k\}_{k\in I} \cap Q_h(x))$$

respectively, where $\natural$ denotes the cardinality of a set. The upper and lower Beurling densities of $\{\lambda_k\}_{k\in I}$ are defined as

$$D^+(\{\lambda_k\}_{k\in I}) = \limsup_{h\to\infty} \frac{v^+(h)}{h^d} \text{ and } D^-(\{\lambda_k\}_{k\in I}) = \liminf_{h\to\infty} \frac{v^-(h)}{h^d}.$$

Now we will introduce the concept of relatively separated sequences.

Definition 3.5 (Relatively Separated and Separated Sequences). Let I be a countable index set and $\{\lambda_k\}_{k\in I} \in \mathbb{R}^d$. We say that

(i) a point $\lambda \in \mathbb{R}^d$ is an accumulation point for $\{\lambda_k\}_{k\in I}$ if every open ball in $\mathbb{R}^d$ centered at λ contains infinitely many λ_k,

(ii) $\{\lambda_k\}_{k\in I}$ is separated if $\inf_{j\neq k} |\lambda_j - \lambda_k| > 0$ and there exists a constant $\delta > 0$ such that $|\lambda_j - \lambda_k| \geq \delta$ for all $j \neq k$ is called a separation constant, and

(iii) $\{\lambda_k\}_{k\in I}$ is relatively separated if it is a finite union of separated sequences.

Using the concepts of Beurling density and relatively separated sequences, we can see how these concepts are related.

Lemma 3.6 (Lemma 9.1.3 in [1]). *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{R}^d$. Then the following are equivalent:*

(i) The upper Beurling density is finite i.e, $D+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$.

(ii) For some (and therefore every) $h > 0$ there is a natural number N_h such that each cube $Q_h(hn)$, $n \in \mathbb{Z}^d$, contains at most N_h points from $\{\lambda_k\}_{k\in\mathbb{Z}}$, i.e.,

$$\sup_{n\in\mathbb{Z}^d} \natural(\{\lambda_k\}_{k\in\mathbb{Z}} \cap Q_h(hn)) < \infty.$$

(iii) $\{\lambda_k\}_{k\in\mathbb{Z}}$ *is relatively separated.*

Proof. To show that (i) implies (ii) suppose the upper Beurling density $D^+(\{\lambda_k\}_{k\in\mathbb{Z}})$ is finite. Then there exists some constant N such that for large $h > 0$,

$$\frac{v^+(h)}{h^d} \leq N.$$

This satisfies (ii) for large values of h which consequently holds for all $h > 0$.

To show (ii) implies (iii), choose $h > 0$ such that (ii) is satisfied. Then we will determine how $\{\lambda_k\}_{k\in\mathbb{Z}}$ can be split into h-separated sequences. Let $e_1, ..., e_{2^d}$ denote the vertices of the unit cube $[0,1]^d$, and consider the sets

$$Z_j = (2\mathbb{Z})^d + e_j,$$

where $j = 1, ..., 2^d$. Then $\mathbb{Z}^d$ is the disjoint union of the sets $Z_1, ..., Z_{2^d}$. Since $\{Q_h(hn)\}_{n\in\mathbb{Z}^d}$ is a disjoint cover of $\mathbb{R}^d$, this means that $\mathbb{R}^d$ is the disjoint union of the sets

$$B_j = \bigcup_{n\in Z_j} Q_h(hn),$$

The cubes $Q_h(hn)$ which make up a given B_j have a distance of at least h, i.e, the distance between any two arbitrary elements in $Q_h(hn)$ and $Q_h(hm)$ is at least h for $n \neq m$. From our assumption, each of these cubes $Q_h(hn)$ contain at most N_h elements from our sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$. This also means $\natural(\{\lambda_k\}_{k\in\mathbb{Z}} \cap Q_h(hn)) \leq N_h$. Since

$$\{\lambda_k\}_{k\in\mathbb{Z}} \cap B_j = \bigcup_{n\in Z_j} (\{\lambda_k\}_{k\in\mathbb{Z}} \cap Q_h(hn)),$$

we can split $\{\lambda_k\}_{k\in\mathbb{Z}} \cap B_j$ into N_h sets which are h-separated. Thus $\{\lambda_k\}_{k\in\mathbb{Z}}$ can be split into $2^d N_h$ sequences which are h separated.

To show (iii) implies (i), assume that our sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ is relatively separated. This means that we can choose a partition of $\{\lambda_k\}_{k\in\mathbb{Z}}$ into a finite number of sequences, which we will denote as $\Lambda_1, \Lambda_2, ..., \Lambda_r$ where each Λ_k is separated with separation constant $\delta_k < 1$. Let

$$\delta := \min\Big\{\frac{\delta_1}{\sqrt{d}}, \frac{\delta_2}{\sqrt{d}}, ..., \frac{\delta_r}{\sqrt{d}}\Big\}.$$

Then the maximum distance between two points in the same cube is

$$\sqrt{\delta^2 + ... + \delta^2} = \sqrt{d\delta^2} = \sqrt{d}\delta = \frac{\sqrt{d}\delta_k}{\sqrt{d}} = \delta_k,$$

where $\delta_k = \min(\delta_1, \ldots, \delta_r)$. Thus any cube $Q_\delta(x)$ will contain at most one point from each Λ_k and therefore at most r points from $\{\lambda_k\}_{k\in\mathbb{Z}}$. This implies that

$$D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) = \lim_{h\to\infty} \sup \frac{v^+(h)}{h^d} = \lim_{h\to\infty} \sup \frac{h\delta}{(h\delta)^d} \leq \lim_{h\to\infty} \sup \frac{r(h+1)^d}{(h\delta)^d} = \frac{r}{\delta^d} < \infty,$$

completing the proof. ■

Combining these results leads us to a rather sad result for frames of translates:

Theorem 3.7 (Theorem 9.6.1 in [1]). *A system of the form $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is never a frame for $L^2(\mathbb{R})$, regardless of the choice of the function $\phi \in L^2(\mathbb{R})$ and sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$.*

To prove this we will be proving two separate statements which amount to the existence of one frame bound implying the other is either 0 or infinity respectively.

(i) If $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) = \infty$, then $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is not a Bessel sequence.

(ii) If $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$, then $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ then the lower frame condition fails.

Proof. To prove statement (i), recall from lemma 3.6 that $D^+(\{\lambda_k\}_{k\in\mathbb{Z}}) < \infty$ is equivalent to $\{\lambda_k\}_{k\in\mathbb{Z}}$ being a relatively separated sequence. We will assume that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is not relatively separated so we must show that $\{T_{\lambda_k}\phi\}$ is not a Bessel sequence. Consider the function $x \mapsto \langle\phi, T_x\phi\rangle$, $x \in \mathbb{R}$. Since the function is continuous by lemma 3.2 and nonzero for $x = 0$, there exists an open interval $(-h, h)$, $h > 0$, such that

$$\mu := \inf_{x\in(-h,h)} |\langle\phi, T_x\phi\rangle| > 0.$$

Consider an arbitrary $N \in \mathbb{N}$. By lemma 3.6, for $a \in \mathbb{R}$, there exists an interval $(a-h, a+h)$ which contains at least N elements from the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$. Define

$$\Lambda_N := \{k \in \mathbb{Z} : \lambda_k \in (a-h, a+h)\} = \{k \in \mathbb{Z} : \lambda_k - a \in (-h, h)\}.$$

Then we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle T_a\phi, T_{\lambda_k}\phi\rangle|^2 &\geq \sum_{k\in\Lambda_N} |\langle T_a\phi, T_{\lambda_k}\phi\rangle|^2 \\
&= \sum_{k\in\Lambda_N} |\langle T_a\phi, T_{\lambda_k - a}\phi\rangle|^2 \\
&\geq N\mu^2 \\
&= \frac{N\mu^2}{||\phi||^2}||T_a\phi||^2.
\end{aligned}$$

Since our choice of N was arbitrary, it follows that $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ is not a Bessel sequence in $L^2(\mathbb{R})$, proving statement (i).

To prove (ii), let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ and assume that $D^+(\Lambda) < \infty$. We need to show that $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ does not satisfy the lower frame bound condition of $A > 0$. By lemma 3.6, $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a finite union of separated sets, so we can write it as

$$\{\lambda_k\}_{k\in\mathbb{Z}} = \bigcup_{j=1}^{s}\{\lambda_k\}_{k\in I_j},$$

where each set $\{\lambda_k\}_{k\in I_j}$ is separated. Choose a separation constant $\delta > 0$ for each sequence $\{\lambda_k\}_{k\in I_j}$, $j = 1, ..., s$, and consider $h \in \left(0, \frac{\delta}{2}\right)$. With $I := [-h, h]$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle \chi_I, T_{\lambda_k}\phi\rangle|^2 &= \sum_{j=1}^{s}\sum_{k\in I_j} |\langle \chi_I, \chi_I T_{\lambda_k}\phi\rangle|^2 \\
&\leq \sum_{j=1}^{s}\sum_{k\in I_j} ||\chi_I||^2||\chi_I T_{\lambda_k}\phi||^2. \qquad (3.0.1)
\end{aligned}$$

By the choice of I, the set of intervals $\{I - \lambda_k\}_{k\in I_j}$ are disjoint. Define

$$\Delta_j := \bigcup_{k\in I_j}(I - \lambda_k).$$

We have

$$\sum_{k\in I_j}||\chi_I T_{\lambda_k}\phi||^2 = \sum_{k\in I_j}\int_I |\phi(x-\lambda_k)|^2 dx = \int_{\Delta_j}|\phi(x)|^2 dx.$$

Thus via equation 3.0.1,

$$\sum_{k\in\mathbb{Z}}|\langle\chi_I, T_{\lambda_k}\phi\rangle|^2 \leq ||\chi_I||^2\sum_{j=1}^{s}\int_{\Delta_j}|\phi(x)|^2 dx.$$

Applying Lebesgue's dominated convergence theorem shows that for each fixed $j = 1, ..., s$,

$$\int_{\Delta_j}|\phi(x)|^2 dx \to 0 \text{ as } h \to 0.$$

Thus $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ does not have a lower frame bound in $L^2(\mathbb{R})$. Since our frame $\{T_{\lambda_k}\phi\}_{k\in\mathbb{Z}}$ cannot have an upper and lower frame bound simultaneously, we conclude that there is not such frame construction that is a frame of translates for $L^2(\mathbb{R})$. ∎

The proof above shows that when the upper Beurling density D^+ is infinite, then the upper frame condition does not hold. Now when the upper Beurling density D^+ is finite, the lower frame condition fails. Since a frame condition fails in both settings, we conclude that there is no such frame of translates for $L^2(\mathbb{R})$. Thus proving Theorem 3.7, our main result.

Chapter 4

Further Research

There are many different ways that one can continue to pursue frame theory. There are many nice papers to begin with depending on the topic you are interested in. If we are looking for a historical perspective then we can begin with the paper which is said to be the genesis of frame theory written by RJ Duffin and AC Schaeffer [6]. Their paper is the first one to define a frame explicitly.

Another area would be to continue looking at frames of translates. As seen, frames of translates exist for subspaces of $L^2(\mathbb{R})$. In the paper by Casazza, Christensen and Kalton [4], they attempt to give different conditions on the types groups which would admit frames of translates.

If we want to go in a more applied direction we can look at the papers by Dennis Gabor and Bernier, David, and Taylor [7, 9]. Gabor's paper talks about communication theory and goes into one of the types of frame constructions mentioned at the beginning of our section on infinite dimensional frames, called the Gabor frame. In [9]the topics of bandlimited spaces and the wavelet frame construction are discussed in different settings.

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