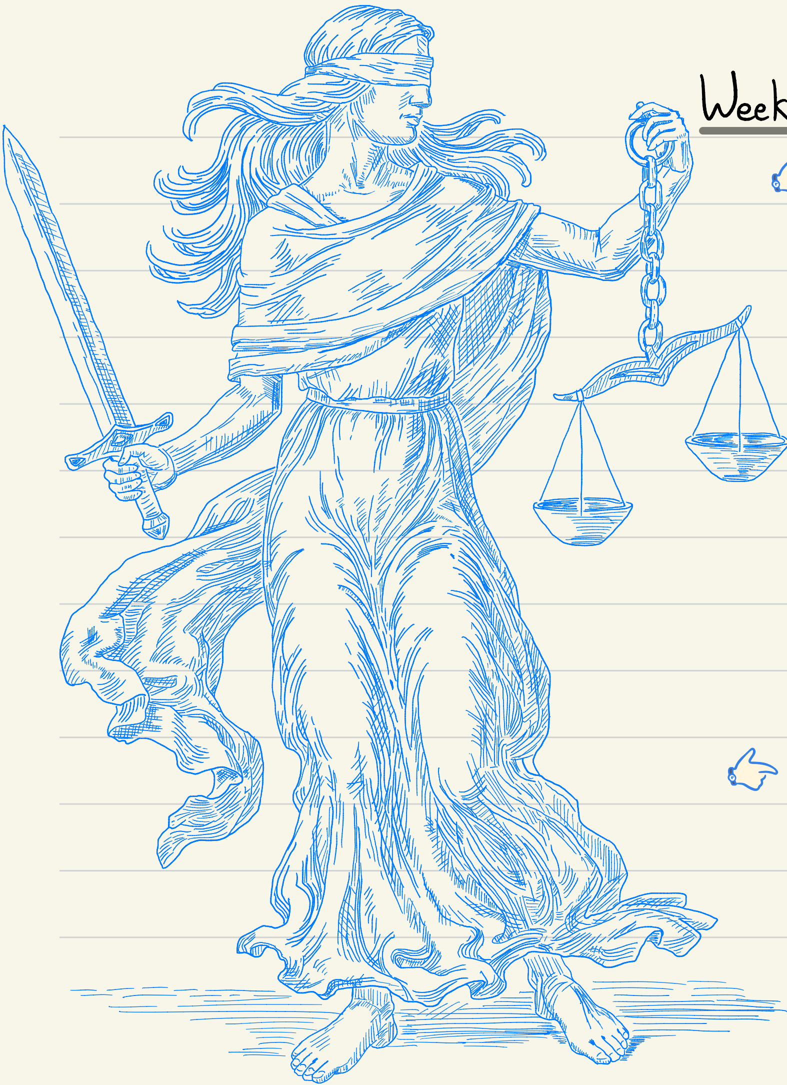
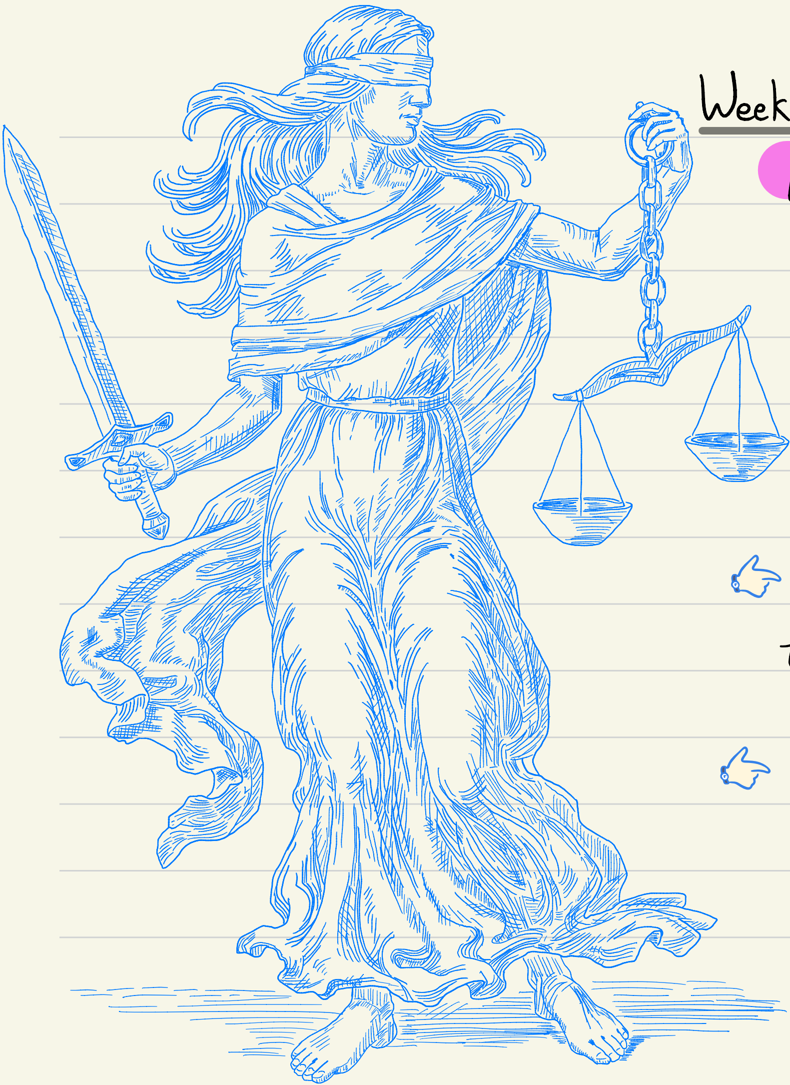


# Week 10) Section 7.2 - Scalar Surface Integrals

- A natural extension of Week 7 on 2D Change of Variables!

- Old material I'll reference/review today:
  - Double Integration to find areas.
  - 2D CoVs and all those approximations I skipped!
  - Scalar Line Integrals. (w.r.t. arclength)
  - Parametric Equations of surfaces.

- We'll use bits and pieces of all these ideas to develop yet another multivariable integration idea.



# Week 10) Section 7.2 - Scalar Surface Integrals

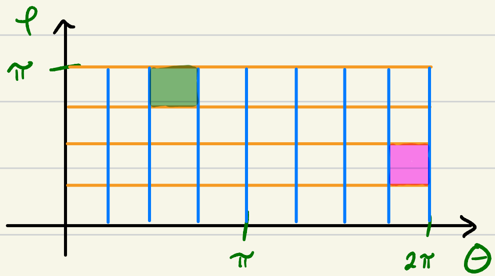
## Learning Outcomes

- Find areas of curvy surfaces in 3D. (Like going from Calc I integrals to scalar line integrals!)
- Determine some physical interpretations of these integrals.
- We'll end with a math fun fact!

## Warm-up

Parameterize the portion of the sphere of radius $R$ which lies on and above the $xy$-plane.

**To get entire sphere:**

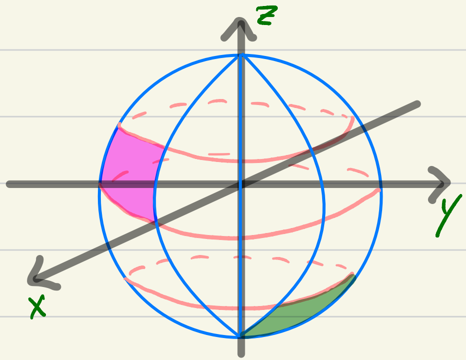



$\vec{x}(\theta, \varphi)$

$x = R\cos\theta\sin\varphi$

$y = R\sin\theta\sin\varphi$

$z = R\cos\varphi$




**To get hemisphere:**

Restrict the domain!

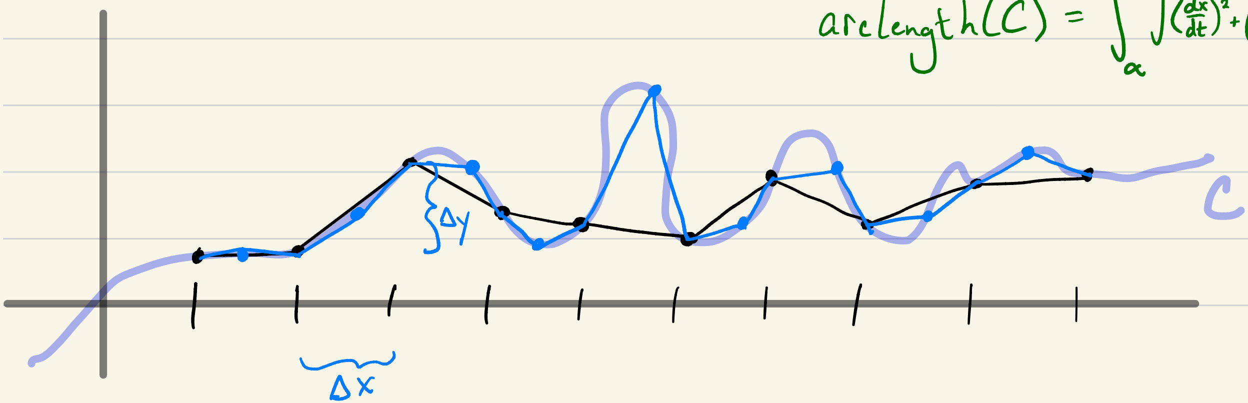
$$0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le \varphi \le \frac{\pi}{2}$$

**Question:** How do we integrate over the surface like the sphere on the last slide?

How did we develop the integrals for calculating arc length?

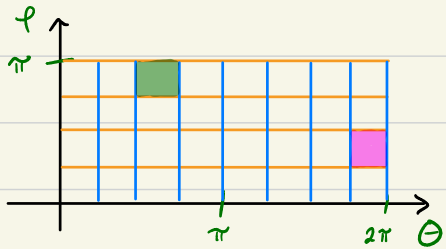
- Start with an approximation
- Refine the approximation
- Take limits

$$\text{arclength}(C) = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



**Question:** How do we integrate over the surface like the sphere on the last slide?

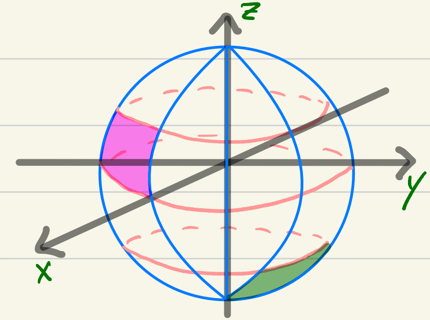
We'll solve this for parametric surfaces.




parametric space

$$\vec{x}(\theta, \varphi)$$

$$x = R\cos\theta\sin\varphi$$
$$y = R\sin\theta\sin\varphi$$
$$z = R\cos\varphi$$



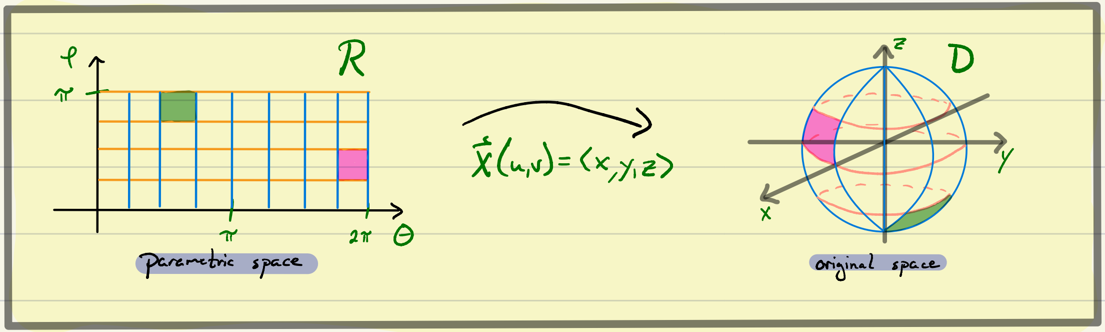

original space

This will be similar to 2D CoV (Week 7) in that we compute in parametric space, but want the answer in our original space.

# The Pesky Details (Continuation from Week 7!)

⚠ More important to pay attn and follow what is happening than to write it down!

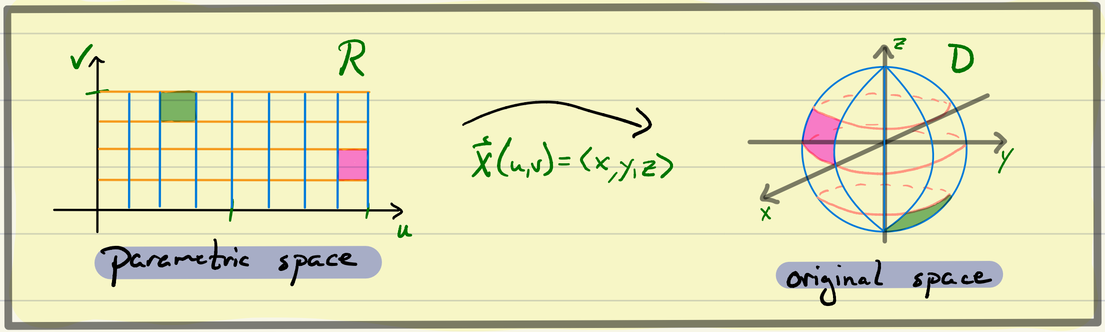
1) Find a parameterization [⚠ not nec. linear] and a "nice" region $R$ such that $\vec{x}(R) = D$ in an injective (1:1) manner ⊛




⊛ Injective is important so that parts of $D$ don't get covered twice, but it's okay if it's just on the boundary since that's a line segment (zero area).

# The Pesky Details

1) Find a parameterization [⚠ not nec. linear]
and a "nice" region $R$ such that
$\vec{x}(R) = D$ in an injective (1:1) manner ⊛

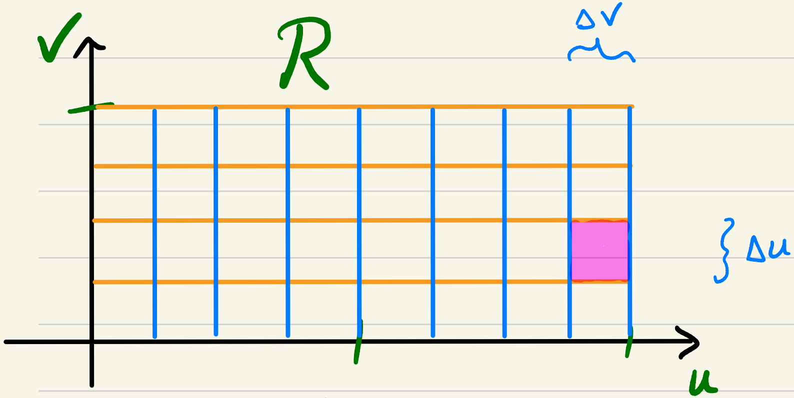



2) Add up the volumes of boxes over "really curvy" rectangles.

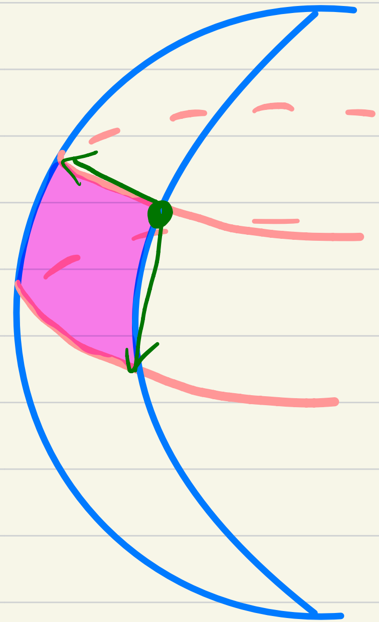
$$\iint_D f \, dS \approx \sum f\left(\begin{smallmatrix}\text{sample}\\ \text{point}\end{smallmatrix}\right) \cdot \underbrace{\text{Area}\left(\begin{smallmatrix}\text{Curvy}\\ \text{rectangles}\end{smallmatrix}\right)}_{\text{so how do we find the area of curvy rectangles?}}$$

Integration w.r.t. surface!

# The Main Idea



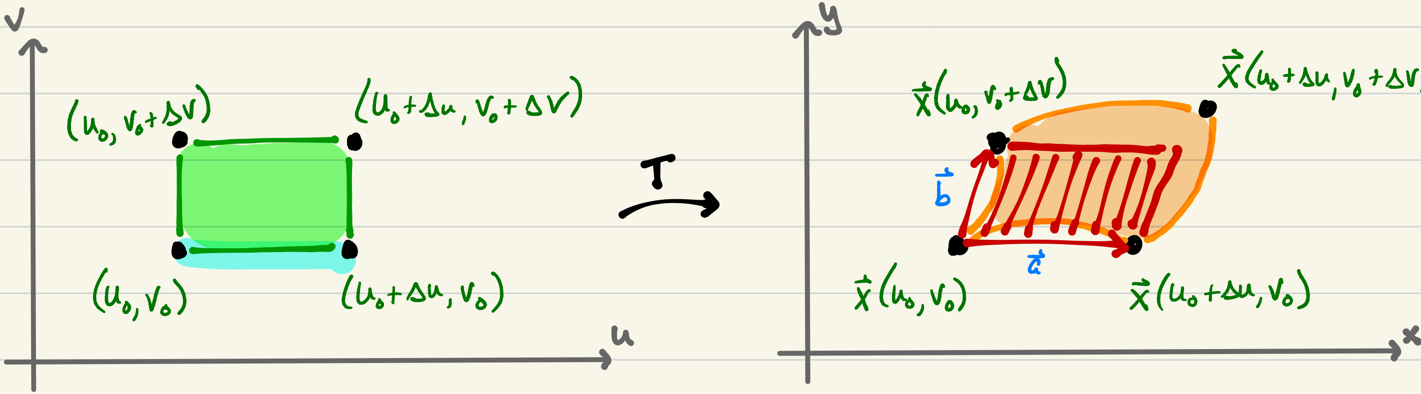

Parametric space



original space

- Fix a corner.
- Draw vectors to adjacent corners.
- What do two vectors describe geometrically?

# Area of Curvy Rectangles




**Define:**

$$\vec{a} = \vec{x}(u_0+\Delta u, v_0) - \vec{x}(u_0, v_0)$$

$$\vec{b} = \vec{x}(u_0, v_0+\Delta v) - \vec{x}(u_0, v_0)$$

**Key Idea:** As $\begin{matrix}\Delta u \to 0 \\ \Delta v \to 0\end{matrix}$, the area of ∥'gram spanned by $\vec{a}$ & $\vec{b}$ approaches the area of curvy rectangles.

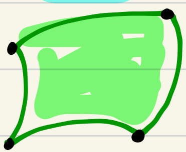
⚠ $\vec{a}, \vec{b}$ hard to work with directly, so as ~~lazy~~ efficient mathematicians, make another approx.!

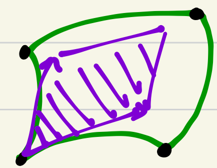
$$\frac{\vec{a}}{\Delta u} = \frac{\vec{x}(u_0 + \Delta u, v_0) - \vec{x}(u_0, v_0)}{\Delta u} \longrightarrow \vec{x}_u(u_0, v_0)$$

$$\frac{\vec{b}}{\Delta v} = \frac{\vec{x}(u_0, v_0 + \Delta v) - \vec{x}(u_0, v_0)}{\Delta v} \longrightarrow \vec{x}_v(u_0, v_0)$$

So for small $\Delta u, \Delta v$: $\vec{a} \approx \vec{x}_u \cdot \Delta u$ & $\vec{b} \approx \vec{x}_v \cdot \Delta v$

In pictures: $\text{Area} \approx |\vec{a} \times \vec{b}| \approx |\vec{x}_u \Delta u \times \vec{x}_v \Delta v| = |\vec{x}_u \times \vec{x}_v| \, \Delta u \, \Delta v$





The missing factor!

$\Delta A$ in uv-space!

⚠ In an Adv. Calculus class, you would likely prove all these approx. work!

Thus by taking limits $\Delta u \to 0$, $\Delta v \to 0$, we obtain the following formula:

$$\boxed{\iint_D f(x,y,z)\, dS = \iint_R f(\vec{x}(u,v))\, |\vec{x}_u \times \vec{x}_v|\, dA.}$$

Surface scalar integral

Double integrals we already know and love!

**Notation:**

Here we will always write the cross-product, as the associated Jacobian (derivative matrix) is not square! So the determinant does not exist.

Thus by taking limits $\Delta u \to 0$, $\Delta v \to 0$, we obtain the following formula:

$$\boxed{\iint_D f(x,y,z)\, dS = \iint_R f(\vec{x}(u,v))\, |\vec{x}_u \times \vec{x}_v|\, dA.}$$

Surface scalar integral

Double integrals we already know and love!

Remark:

If $z=0$, then this is identical to our 2D CoVs formula from week 7, and the details are the same string of approximations!

# Surface Integrals of Scalar Valued Functions

**Def$^n$:** Let $f: S \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a continuous scalar function.

Assigns a value to a point on the surface.

Let $\vec{x}(u,v): D \to S$ be a parameterization of $S$. Then the scalar surface integral is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{x}(u,v))\, \|\vec{x}_u \times \vec{x}_v\|\, dA$$

Capital for surface integral

Q: What relationship does this have to the surface?

**Remark:**

- No way to calculate LHS except through double integral on RHS.
- This is independent of parameterization, thus we often write

$$\iint_S f\, dS = \iint_{\vec{x}} f\, dS$$

# Surface Integrals of Scalar Valued Functions

## Physical Interpretations

- In first example, can find surface areas in $\mathbb{R}^3$.

- Let $f(x,y,z)$ give the quantity of something on the surface (amount of snow, yearly rainfall, average temperature, etc...)

- On a smaller scale, if $f(x,y,z)$ is the amount of electrical charge (charge density), then $\iint_S f\, dS$ gives total charge. Similarly integrating surface/area/superficial density gives mass.

- On a more fun scale, we could calculate how much glaze goes on a donut (group work)

Example:

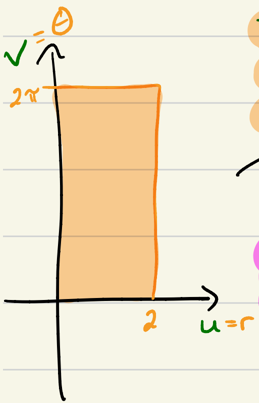
Find the surface area of the paraboloid $z = x^2 + y^2$ with $0 \le z \le 4$.

## Step 1: Parameterize the surface.

$x = r\cos\theta$

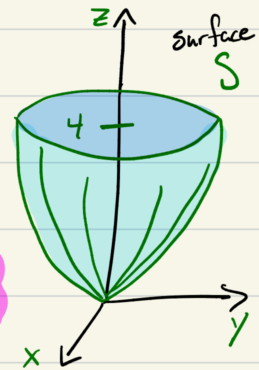
$y = r\sin\theta$

$z = r^2$




$\vec{x}(u,v)$

still cts, 1:1 (except possibly on "endpoints" " boundary)




## Step 2: Calculate area of "curvy rect."

$$\|\vec{x}_r \times \vec{x}_\theta\| = \|(\cos\theta, \sin\theta, 2r) \times (-r\sin\theta, r\cos\theta, 0)\| = \ldots = \sqrt{r^2 + 4r^4}$$

## Step 3: Integrate!

$$\text{Area}(S) = \int_0^{2\pi}\int_0^2 \|\vec{x}_r \times \vec{x}_\theta\|\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\sqrt{1+4r^2}\, dr\, d\theta \qquad \text{(u-sub)}$$

$$= \pi/6\,(17^{3/2} - 1)$$

# Surface Integrals of Scalar Valued Functions

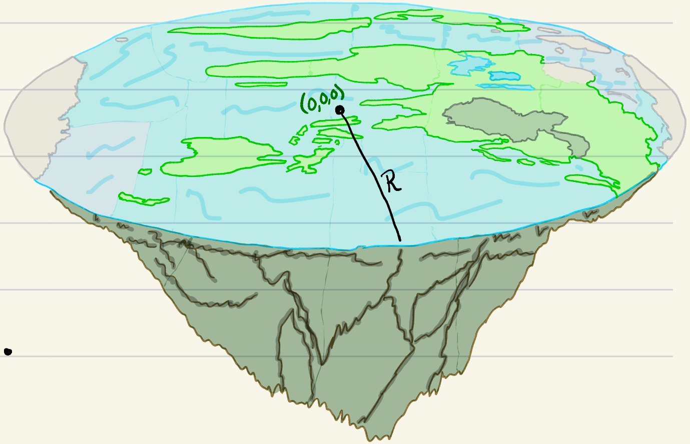
**Example:** Let $S$ be the surface of the Earth. (for simplicity, assume radius of Earth is $R$ inches)

Let $f(x,y,z)$ give the depth of snow at the point $(x,y,z)$ in inches. How can we find the total amount of snow on the surface of the Earth?

For "flat-Earthers"

$z=0$ and this is just

$$\iint_S f(x,y)\,dA = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f(x,y)\,dy\,dx = \int_0^{2\pi}\int_0^R f(r,\theta)\, r\, dr\,d\theta.$$



# Surface Integrals of Scalar Valued Functions

**Example:** Let $S$ be the surface of the Earth. (for simplicity, assume radius of Earth is $R$ inches)


**For "real-Earthers"**

Total amount of snow is given by

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{n} \begin{bmatrix} \text{Area of small} \\ \text{curvy rectangle} \end{bmatrix} \cdot f(x(u_i,v_j), y(u_i,v_j), z(u_i,v_j)).$$

As usual, by taking $n \to \infty$

$$\iint_D \|\vec{x}_u \times \vec{x}_v\| \, f(\vec{x}(u,v)) \, du \, dv$$ yields the exact amount of snow.

Domain of parameterization

# Fun Math Fact

## Borsuk-Ulam theorem

Every continuous function from an $n$-sphere into Euclidean $n$-space ($f: S^n \to \mathbb{R}^n$) maps some pair of antipodal points to the same value.

Examples:

- When $n=1$ ($S^n$ is a circle): There are antipodal pts on the equator that are the same temperature.

This is the 2-sphere

- When $n=2$, two antipodal points with same temp and barometric pressure!

# Surface Integrals of Scalar Valued Functions

**Example:** Let $S$ be the cylindrical surface defined by $x^2+y^2=1$ and $-1 \le z \le 1$. The surface density of $S$ is given by $\mu(x,y,z) = z^2(x^2+y^2)$. Find the mass of $S$.

**Step 1:** Parameterize the surface.

$$x = 1\cos\theta \qquad y = 1\sin\theta \qquad z = t$$

$$0 \le \theta \le 2\pi \qquad -1 \le t \le 1$$

**Step 2:** Calculate area of "curvy rect."

$$\|\vec{x}_\theta \times \vec{x}_t\| = \|\langle -\sin t, \cos t, 0\rangle \times \langle 0,0,1\rangle\| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{vmatrix} = \|\langle \cos\theta, -\sin\theta, 0\rangle\| = 1.$$

**Step 3:** Integrate!

$$\iint_S \mu \, dS = \int_0^{2\pi}\int_{-1}^{1} \mu(\vec{x}(\theta,t))\|\vec{x}_\theta \times \vec{x}_t\| \, dt \, d\theta$$

$$= \int_0^{2\pi}\int_{-1}^{1} t^2 \, dt \, d\theta$$

simple integration

$$= \frac{4}{3}\pi$$