Challenge Problem #4: For those interested in the proof of the Euclidean algorithm, the steps are layed out below. Prove each of the lemmas and combine them at the end to show the Euclidean algorithm produces the GCD of two numbers as the last non-zero remainder.

Lemma 1. *Let a, b, m, and n be integers. If $c \mid a$ and $c \mid b$, then $c \mid (ma + nb)$.*

Proof. Prove directly using the definition of divisibility. □

Lemma 2. *Let a, b, c be integers. Then $\gcd(a + cb, b) = \gcd(a, b)$.*

Proof. Use Lemma 1 to show that the common divisors of the pair $(a + cb, b)$ and the pair (a, b) are exactly the same. □

Theorem 1. *Let $r_0 = a$ and $r_1 = b$ be integers such that $a \geq b > 0$. If the division algorithm is successively applies to obrain $r_j = r_{j+1}q_{j+1} + r_{j+2}$, with $0 < r_{j+2} < r_{j+1}$ for $j = 0, 1, 2, \ldots, n - 2$ and $r_{n+1} = 0$, then $\gcd(a, b) = r_n$, the last non-zero remainder.*

Proof. First show that the algorithm eventually terminates with a zero remainder. Next show that the process preserves the GCD. Conclude the Euclidean algorithm works as claimed.

□

NOTE: Correctly completing the proofs for Lemma 1, Lemma 2, and Theorem 1 is sufficient to achieve a grade of "Meets Expectations", to obtain a grade of "Exceeds Expectation", also prove the following theorem.

Theorem 2. *The greatest common divisor of the integers a and b, not both 0, is the least positive integer that can be written as $ax + by$ for integers x, y.*

Proof. You will need to apply the well-ordering property for natural numbers, which states:

`Every non-empty set of natural numbers has a least element.`

Proceed by defining an appropriate non-empty set of natural numbers. Call the set's least element d. Show that d divides both a and b, so it is a common divisor. Show that every other divisor of both a and b divides d, so it is the greatest of all common divisors.

This will conclude your proof.

□

Grading Criteria This challenge problem will be graded on mathematical correctness of your proofs as well as writing quality. Proofs should be properly formatted and use complete sentences. Mathematics should be smoothly incorporated into the sentence structure and read naturally. Two-column proofs should not appear. Feel free to contact me with specific questions.

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