# DISCRETE FRAMES FOR HIGH-DIMENSIONAL DATA: CONSTRUCTIONS ON REGULAR AND IRREGULAR DOMAINS

by

Kristopher G. Hollingsworth

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences

2020

# Acknowledgements

First and foremost, I would like to thank my academic advisors, Mahya Ghandehari and Dominique Guillot. It has been a pleasure learning from you both, and I appreciate your patience, advice, and continuous support and guidance throughout my thesis work. Thank you both for always challenging me to be better. I would also like to thank the many faculty members in the department who supported me during this process and introduced me to many new ideas and new areas of mathematics; you all created an amazing academic environment where my curiosity could thrive. In particular, I would like to thank Mahya Ghandehari, Sebastian Cioabă, and Felix Lazebnik. Due to the unexpected death of my father, I had to take a leave of absence during my first year of graduate school; without your support, understanding, and consideration, I might not have returned to continue my education in the wake of such a sudden loss. You all have my most sincere gratitude.

In my efforts to improve my teaching while at Delaware, I owe much to Chad Giusti, Eirini Kilikian, and Shannon Miller for their many conversations on the subject and their willingness to share ideas and stories for and from the classroom. Your support and advice during my job search was also invaluable. I do not think I would have survived the process on top of a global pandemic (and everything else) without your help. Many thanks also go to Miss Debby and Pancho: I wish you were still with us so that I could tell you of the impact you had on my life while at Delaware. I know I speak for all the graduate students, past and present, when I say you are greatly missed. You both provided insight and wisdom in every conversation; you were the best academic support team any graduate student could wish for.

To my academic cohort, thank you for your friendship and camaraderie throughout these years. To Hasan, Shukai, Ben, Greg, and Sam—I'm glad we all got to be sardines together during our first year, it would never have happened in the new world of social distancing. I also appreciate having had the opportunity to share this journey with undergraduate collaborators on two summer research projects: Paige Matthews and Nathaniel Kim. Working with each of you was a pleasure, and Chapter 5 would certainly have been different without the many conversations I had with you both regarding the material contained therein. I would also like to thank Gautam Aishwarya for always being willing to discuss esoteric math problems and random connections between ideas. I will miss being your office neighbor and taking road trips to conferences together. I hope to see you at one soon, though! My particular thanks to Samuel Eskridge Cogar for his friendship, advice, and bocce skills. I could think of no better teammate for the annual department tournament, and I appreciate all our many conversations from Sobolev spaces to Star Wars and everything in between.

To Robin Moore, it is hard to believe it was ten years ago when you suggested I pursue a degree in mathematics. Thank you for opening up the possibility in my mind, as it was something I had never even considered before I met you. I owe a debt of gratitude to several other professors from my undergraduate studies as well. To Fred Hickling, Steve Butcher, and Ramesh Garimella, thank you for preparing me for the rigors of an advanced degree. Without the many hours spent in your offices asking questions, I doubt I would have had the necessary foundation to pursue my current degree.

To my family, thank you for your continued love and support. Knowing you are always in my corner has made getting through some of the rougher days on this journey a little easier. Finally, to Kathleen, words cannot adequately express how much your love and support has meant over the past five years, and even more so over the past few months while writing this thesis. I could not ask for a more patient and understanding partner, and I look forward to starting our new life together, wherein neither of us are graduate students. It should be quite an adventure!

# Table of Contents

# List of Tables

# List of Figures

# Abstract

The theory of discrete frames was introduced in the 1950s by Duffin and Schaeffer, when they initiated a systematic study of dictionaries for the efficient and robust representation of data. This work was generalized by Daubechies, Grossmann, and Meyer in the late 1980s. As the representations in a Hilbert space provided by a basis often fail to provide sufficient flexibility, their work led to an explosion of interest in the development of discrete frame theory, persisting to this day. Applications range from the analysis, compression, and transmission of data to the study of function spaces. The primary results for high-dimensional spaces are existence theorems and generalizing known constructions of frames to other data sets.

The main theme of this thesis is the systematic construction of discrete frame systems in two distinct settings. In the first part, we develop Gabor-type frames for functions defined on the vertex set of a graph. We propose a general framework which captures many of the existing discrete frame constructions in this setting. We also provide necessary and sufficient conditions on the choices of translations and localizing function such that they lead to Gabor-type frames. Sharp frame bounds are calculated for several existing cases, and a general formula for finding them is provided. We conclude this part with a study of frames for Cayley graphs, where tools from representation theory for finite (not necessarily abelian) groups can be applied to simplify many operations.

In the second part of this thesis, we develop wavelet frames through sampling continuous reproducing formulas for functions defined on the regular domain of $\mathrm{M}_n(\mathbb{R})$.

The work in this context relies heavily on the theory of square-integrable representations of locally compact groups, but all necessary background is provided. In particular, we generalize a previous construction of Ghandehari, Syzdykova, and Taylor for $L^2(\mathrm{M}_2(\mathbb{R}))$ to $L^2(\mathrm{M}_n(\mathbb{R}))$ for any dimension $n$. We also significantly improve the frame bounds for these constructions, improving the numerical stability of the frame design. We conclude this part with the specific details for the case $n = 2$ to show how the general theory can be applied to construct discrete frames.

# Chapter 1

# INTRODUCTION

Given the increasing amount of data being recorded, stored, transmitted, and analyzed since the advent of modern computing and telecommunications, there has been a corresponding increase in research related to finding efficient, robust methods for representing information digitally (*i.e.*, discretely). *Discrete frame theory* provides a mathematical framework for studying infinite series representations of functions in a Hilbert space for stable and possibly redundant representations of data. In many applications of signal processing, frames provide a better representation than an orthonormal basis. For example, certain frame representations are more resistant to additive noise (sending information over a noisy channel) than orthonormal basis representations. The redundant nature of frames enables reconstruction even if part of the signal is lost or corrupted. Unsurprisingly, the underlying mathematical structure of many wireless communication networks (*e.g.*, CDMA and GPS) relies on frame theory, with other applications ranging from medical data analysis to the study of function spaces.

Informally, a discrete frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\left\{\phi_x\right\}_{x\in X}$, where $X$ is some countable index set, and such that each function $f \in \mathcal{H}$ can be reconstructed from the sequence $\left\{\langle f, \phi_x \rangle\right\}_{x\in X}$. While this is similar to the definition of a basis, frames remove the constraint requiring linear independence, while maintaining spanning and adding stability requirements. In particular, the frame coefficients satisfy

$$A\,\|f\|_{\mathcal{H}}^2 \le \sum_{x\in X} |\langle f, \phi_i \rangle|^2 \le B\,\|f\|_{\mathcal{H}}^2 \tag{1.1}$$

for some positive constants $A$ and $B$. One important consequence of the definition for discrete frames is worth noting immediately: if the frame is not a (Riesz) basis for $\mathcal{H}$, then there is some measure of redundancy naturally built into the representation. A historical review and technical introduction to discrete frame theory is given in Chapter 3.

The vectors constituting a frame do not need to be related in any way. However, researchers have been greatly interested in frames which can be implemented by the application of an easily described set of discrete actions applied to a single vector, called the *analyzing* or *mother wavelet*. In the present thesis, we construct two particular classes of frames: Gabor frames and wavelet frames. *Gabor frames* are given by applying modulation and translation operators to the mother wavelet, which we construct for signals defined on the irregular domain of the vertex set of a graph. Similarly, *wavelet frames* are given by applying dilation and translation operators to the mother wavelet, which we construct for signals defined on high-dimensional Euclidean spaces.

**Gabor-type Frames for Graphs.** We first study Gabor-type frames for signals defined on a graph. Let $\Gamma$ be a fixed graph with vertex set $V(\Gamma) = \{1, 2, \ldots, N\}$. A signal on $\Gamma$ is a function $\mathfrak{f} : V(\Gamma) \to \mathbb{C}$. We identify $\mathfrak{f}$ with the vector $(\mathfrak{f}(1), \mathfrak{f}(2), \ldots, \mathfrak{f}(N))^\top$ in $\mathbb{C}^N$, where $M^\top$ denotes the transpose of the matrix $M$. A natural technique to analyze signals defined on graphs that is rapidly gaining popularity involves fixing a basis of eigenvectors for a chosen matrix associated with the graph and expanding a given signal defined on the graph in that basis. The goal in doing so is to improve signal processing techniques by working with a basis that is more adapted to the graph compared to an arbitrary basis of $\mathbb{C}^N$. Natural examples of matrices associated to the graph include: (1) the adjacency matrix $A_\Gamma$ with entries $(i, j)$ equal to 1 when there is an edge from vertex $i$ to vertex $j$, and 0 otherwise; and (2) the graph Laplacian $L_\Gamma := D_\Gamma - A_\Gamma$, where $D_\Gamma$ is the diagonal matrix with entry $d_{i,i}$ equal to the degree of vertex $i$. Other matrices such as the normalized Laplacian, $D_\Gamma^{-1/2} L_\Gamma D_\Gamma^{-1/2}$, and the random walk Laplacian, $D_\Gamma^{-1}L_\Gamma$, have also been considered.

Our starting point in Chapter 4 is the Gabor-type construction of frames given by Shuman, Ricaud, and Vandergheynst in [103]. First, we propose a general framework for constructing Gabor-type frames for signals on graphs. Our approach uses general and flexible families of translation operators which include many previous constructions such as those provided in [53, 59, 94, 103]. For each such family of translations, we obtain the sharp frame bounds for the associated frames. We also provide worst-case bounds for arbitrary choices of the localizing function.

In the second part of the chapter, we examine the constructed frames in the special case where the graph $\Gamma$ is a *normal Cayley graph*. In this case, an orthonormal basis of eigenvectors of the adjacency matrix or the graph Laplacian of the graph can be explicitly obtained by exploiting the representation theory of the associated group [6]. Using this basis, we are able to study properties of our Gabor-type frames and how they relate to the structure of the underlying group: an investigation which initially began in [55].

**Wavelet Frames for $L^2(\mathrm{M}_n(\mathbb{R}))$.** Continuous frames are similar to discrete frames, where we integrate over a continuous index set in place of the summation in (1.1). A well studied question regarding continuous frames is: given a continuous, integral reproducing formula on a Hilbert space, when can one choose a discrete subset of sample points to obtain a discrete frame?

While recent work (described in Section 3.1) has completely characterized when a discretization by sampling is possible, most work on the discretization problem is non-constructive in nature; this leaves the question of *how* to choose the set of sampling points to obtain discrete frames open. As the dimension of the signal space grows, developing suitable and efficient frames becomes increasingly difficult. The easiest method for handling multi-dimensional cases would be to simply use the tensor product of 1-dimensional solutions. However, as we will illustrate later, this approach fails to capture many of the geometric features of high-dimensional signals.

In Chapter 5, we focus solely on the discretization problem of constructing discrete frames by sampling continuous wavelet frames in the orbit of a square-integrable irreducible representation. We seek to improve the results of [58], wherein the theory of square-integrable representations was combined with the geometry of the Euclidean space to construct discrete frames for $L^2(\mathbb{R}^4)$. As in [58], we follow the method developed in [11], where a general framework for the construction of higher dimensional continuous wavelet transforms was investigated. Essentially, if a locally compact group $H$ acts on $\mathbb{R}^n$ in such a manner that $H$ acts freely and transitively on an open subset $\mathcal{O}$ in $\widehat{\mathbb{R}^n}$, then an associated continuous wavelet transform theory can be developed as described in Section 5.2. Various methods, such as careful geometric techniques, can then be used to discretize continuous wavelet transforms, or continuous frames resulting from them, in order to produce discrete frames.

In particular, we use representation theory of the affine group to construct a tight continuous frame. We then obtain a discrete frame through careful geometric techniques for discretizing the reproducing formula. We improve the construction in [58] significantly by reducing the frame condition number (the ratio of optimal frame bounds) from about 1782 to 33 for $L^2(\mathbb{R}^4)$. More importantly, the previous construction was provided only for $L^2(\mathbb{R}^4)$ which we generalize to $L^2(\mathbb{R}^{n^2})$ for any $n \in \mathbb{N}$. Note that square-integrable representations provide us with the only reasonable framework to produce discrete frames, since the existence of a discrete frame in this setting implies the square-integrability of the unitary representation generating the associated continuous frame (see [5] for more details).

**Organization.** The remainder of this thesis is organized as follows. Chapters 2 and 3 survey the most important results from harmonic analysis and frame theory, respectively, that are required in this thesis. These also serve to establish notations that will be used throughout this work. In Chapter 4, we describe a general construction for Gabor-type discrete frames for functions defined on a graph and prove sharp frame bounds. We also study some specific constructions on Cayley graphs and discuss how the group structure can be leveraged to simplify calculations. In Chapter 5, we construct a tight continuous wavelet transform for $L^2(\mathrm{M}_n(\mathbb{R}))$. By carefully considering the geometry of the Euclidean space, we determine a sampling set to yield a discrete frame and compute the relevant frame bounds. We find subclasses of signals for which these bounds can be improved, and we discuss possible extensions to other function spaces. In Chapter 6, we conclude this thesis by discussing some open problems related the discrete frame constructions from Chapters 4 and 5. This discussion includes some partial results.

# Chapter 2

## BACKGROUND AND NOTATIONS: HARMONIC ANALYSIS

This chapter contains the necessary background from harmonic analysis for this thesis. Here we introduce notations and provide fundamental tools used in the subsequent chapters. First, we review the general theory of unitary representations of locally compact groups. Next, we describe the process of inducing representations of subgroups to representations of the whole group. We conclude the chapter with a brief introduction of the general theory of Fourier analysis and its links to representation theory. Additionally, we give concrete examples of the groups to be used in subsequent chapters, demonstrating how the notations from the general theory manifest for our specific settings.

For a thorough introduction to representation theory with applications to harmonic analysis, we recommend the book by Terras [110] for finite groups, and the books of Folland [41] and Hewitt and Ross [71] for more general study. For the topic of induced representations, we refer to the book of Kaniuth and Taylor [78]. For the topic of Fourier analysis as it is used in the present work, we refer to Chapters 3 and 4 of [41] or Chapters 1 and 2 of [93].

## 2.1 Representations of Locally Compact Groups

A *topological group* $G$ is a topological space with a group structure such that the group operations are continuous. That is $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are both continuous functions from $G \times G \to G$ and $G \to G$, respectively. A *locally compact group* is a topological group whose topology is locally compact and Hausdorff. That is, for any two distinct points $x, y \in G$, there exists disjoint, precompact neighborhoods $U, V$ such that $x \in U$ and $y \in V$.

One of the most important properties of locally compact groups is the existence of a unique (up to positive scaling) left-invariant Radon measure [41, Theorem 2.10 & 2.20], called the *left Haar measure* which we denote by $\lambda_G$. For any function $f$ in $C_c(G)$, let $\int_G f(x)\,dx$ denote the integral of $f$ with respect to the measure $\lambda_G$. With integration defined on $G$, we can define $L^p(G)$ in the usual way

$$L^p(G) = \left\{ f : G \to \mathbb{C} \;\middle|\; \int_G |f(x)|^p\,dx < \infty \right\},$$

where two functions in $L^p(G)$ are identified if they are equal a.e.

It is worth noting that, in general, the left Haar measure need not be right-invariant. However, there exists a function $\Delta_G : G \to \mathbb{R}^+$ such that

$$\int_G f(xy)\,dx = \frac{1}{\Delta_G(y)} \int_G f(x)\,dx \tag{2.1}$$

and

$$\int_G f(x^{-1})\,dx = \int_G f(x)\Delta_G(x^{-1})\,dx, \tag{2.2}$$

for any $y \in G$ and $f \in L^1(G)$. In fact, $\Delta_G$ is a continuous homomorphism from $G$ to the multiplicative group $\mathbb{R}^+$. We call the function $\Delta_G$ the *modular function* of $G$, and if $\Delta_G \equiv 1$ then we say $G$ is a *unimodular group*. Several examples of unimodular groups include abelian, discrete, and compact groups. One of the simplest examples of a non-unimodular group is the so-called *ax+b* group, the group of affine transformations on $\mathbb{R}^d$, which we will see in Example 2.1.7.

A *unitary representation* of a locally compact group $G$ is a map $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that $\pi$ is a group homomorphism. Here $\mathcal{H}_\pi$ is some Hilbert space which we usually call the *representation space* of $\pi$, and $\mathcal{U}(\mathcal{H}_\pi)$ is the group of unitary operators on $\mathcal{H}_\pi$. The dimension of $\mathcal{H}_\pi$ is called the *dimension* or *degree* of the representation $\pi$ and will be denoted by $d_\pi$. We say $\pi$ is a *continuous* unitary representation if $\pi$ is

WOT-continuous. That is, for each $u, v \in \mathcal{H}_\pi$, the *coefficient function* defined as the map

$$\pi_{u,v} : G \to \mathbb{C}, \qquad \pi_{u,v}(x) = \langle\, \pi(x)v, u \,\rangle_{\mathcal{H}_\pi}, \tag{2.3}$$

is continuous on $G$. When the group $G$ is finite, one can consider non-unitary representations by replacing the group of unitary operators on $\mathcal{H}_\pi$ with the group of invertible linear operators on $\mathcal{H}_\pi$. However, in the present thesis we only consider unitary representations. Therefore, anytime we refer to representations, we will always mean *unitary* representations.

An important tool in the study of representations is the space of *intertwining operators*. For two representations $\rho$ and $\sigma$ of a group $G$, we define this space as

$$\mathcal{I}(\rho, \sigma) = \left\{ L : \mathcal{H}_\rho \to \mathcal{H}_\sigma \;\middle|\; L \text{ is bounded, linear and } L \circ \rho(x) = \sigma(x) \circ L, \forall x \in G \right\}.$$

We say $\rho$ and $\sigma$ are *unitarily equivalent* if $\mathcal{I}(\rho, \sigma)$ contains a unitary operator.

One last important definition we need to proceed is the notion of an *irreducible representation*. For a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ and a subspace $S$ of $\mathcal{H}_\pi$, we say $S$ is $\pi$-invariant if $\pi(g)S \subseteq S$ for all $g \in G$. Then we call $\pi$ irreducible if the only closed, $\pi$-invariant subspaces of $\mathcal{H}_\pi$ are $\mathcal{H}_\pi$ and the trivial subspace $\{0\}$. The following lemma, due to Issai Schur, classifies irreducible representations in terms of the space $\mathcal{I}(\pi) := \mathcal{I}(\pi, \pi)$, the space of bounded operators which intertwine $\pi$ with itself.

THEOREM 2.1.1 (Schur's Lemma, [41, Theorem 3.5]). *Let $G$ be a locally compact group. Then*

1. *A unitary representation $\pi$ of $G$ is irreducible if and only if $\mathcal{I}(\pi)$ contains only scalar multiples of the identity.*

2. *Suppose $\pi_1$ and $\pi_2$ are irreducible unitary representations of $G$. If $\pi_1$ and $\pi_2$ are equivalent then $\mathcal{I}(\pi_1, \pi_2)$ is one-dimensional, otherwise $\mathcal{I}(\pi_1, \pi_2) = \{0\}$.*

Example 2.1.2 (Regular Representations). An important example of a representation for any locally compact group is the *left regular representation.* This is given by the map $L : G \to \mathcal{U}(L^2(G))$ defined as

$$L(x)f(y) = f(x^{-1}y). \tag{2.4}$$

It is easy to verify that $L$ is a homomorphism. Further, for every $x \in G$, $L(x)$ is an isometry. Indeed, for $f, g \in L^2(G)$, we have

$$\langle\, L(x)f, L(x)g \,\rangle_{L^2(G)} = \int_G f(x^{-1}y)\,\overline{g(x^{-1}y)}\,dy = \int_G f(y)\,\overline{g(y)}\,d(xy) = \langle\, f, g \,\rangle_{L^2(G)}$$

as $\lambda_G$ is left-shift invariant. Clearly, $L(x)^{-1} = L(x^{-1})$, thus $L(x)$ is invertible. As $L(x)$ is an invertible isometry, it is a unitary operator, showing that $L$ is a unitary representation of $G$.

Similarly, the *right regular representation* is given by $R(x)f(y) = \sqrt{\Delta_G(x)}f(yx)$, where $\sqrt{\Delta_G(x)}$ appears so that this representation is unitary with respect to the left Haar measure. That is,

$$\int_G [R(x)f](y)\overline{[R(x)g](y)}\,dy = \int_G \Delta_G(x)\,f(yx)\overline{g(yx)}\,dy = \int_G f(y)\overline{g(y)}\,dy,$$

where the final equality holds by equation (2.1). As with the left regular representation, we have that $R(x)^{-1} = R(x^{-1})$, so $R(x)$ is a unitary operator and $R$ is a unitary representation of $G$. The left and right regular representations are closely related, as demonstrated in the following lemma.

Lemma 2.1.3. *Let $L : G \to \mathcal{U}(L^2(G))$ and $R : G \to \mathcal{U}(L^2(G))$ be defined as above. Define $M : L^2(G) \to L^2(G)$ by $Mf(x) = \dfrac{f(x^{-1})}{\sqrt{\Delta_G(x)}}$. Then:*

1. *$M$ is a unitary operator in $\mathcal{I}(L, R)$, so $L$ and $R$ are unitarily equivalent.*
2. $\left\{\, L(g) \;\middle|\; g \in G \,\right\} \subset \mathcal{I}(R)$ *and* $\left\{\, R(g) \;\middle|\; g \in G \,\right\} \subset \mathcal{I}(L)$.
3. *Unless $G$ is the trivial group with one element, $L$ and $R$ are not irreducible representations.*

*Proof.* The first statement follows from (2.2), as

$$\langle Mf, Mg \rangle_{L^2(G)} = \int_G \frac{f(x^{-1})\overline{g(x^{-1})}}{\Delta_G(x)}\,dx = \int_G \frac{f(x)\overline{g(x)}}{\Delta_G(x^{-1})}\,d(x^{-1}) = \langle f, g \rangle_{L^2(G)}.$$

Combining this with the fact that $M^2 f(x) = [\Delta_G(xx^{-1})]^{-1/2} f(x) = f(x)$, we have that $M^{-1} = M$, showing that $M$ is an invertible isometry, thus unitary. Moreover, $M$ intertwines the representations $L$ and $R$ as

$$M[L(g)f](x) = \frac{L(g)f(x^{-1})}{\sqrt{\Delta_G(x)}} = \sqrt{\Delta_G(g)}\frac{f([xg]^{-1})}{\sqrt{\Delta_G(xg)}} = \sqrt{\Delta_G(g)}Mf(xg) = R(g)[Mf](x).$$

For the second statement, we have for any $h, g, x \in G$,

$$R(g)[L(h)f](x) = \sqrt{\Delta_G(g)}f(h^{-1}xg) = L(h)[R(g)f](x).$$

For the final statement, when $G$ is the trivial group then the identity is the only element. It follows that $L$ and $R$ are each the trivial representation, and therefore they are irreducible. So suppose $G$ is not the trivial group, and note that we have shown $L(g) \in \mathcal{I}(R)$, $R(g) \in \mathcal{I}(L)$. Additionally, $L(g)$ and $R(g)$ are never scalar multiples of the identity unless $g$ is the group identity. It follows immediately from Schur's lemma (Theorem 2.1.1) that $L$ and $R$ are not irreducible representations. □

The *dual object* of a locally compact group $G$ is defined as

$$\widehat{G} = \left\{ [\pi] \;\middle|\; \pi \text{ is an irreducible representation of } G \right\},$$

where $[\pi]$ denotes the equivalence class of irreducible representations which are unitarily equivalent to $\pi$. For abelian groups, every irreducible representation is one-dimensional [41, Corollary 3.6]. This is another immediate consequence of Schur's lemma: for an abelian group $G$ and an irreducible representation $\pi$, we have for any $x, y \in G$ that $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$. So for every $x \in G$, $\pi(x) \in \mathcal{I}(\pi)$. Therefore $\pi(x)$ is a scalar multiple of the identity for every group element $x$. As this would mean every subspace of $\mathcal{H}_\pi$ would be $\pi$-invariant, there must be no non-trivial subspaces. Clearly, this can only occur if $\mathcal{H}_\pi$ is one-dimensional.

Another consequence of Schur's lemma is related to the irreducible representations of compact groups.

THEOREM 2.1.4 ([41, Theorem 5.2]). *If $G$ is compact, then every irreducible representation of $G$ is finite dimensional, and every unitary representation of $G$ is equivalent to a direct sum of irreducible representations.*

Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a representation of a compact group $G$ and let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}_\pi$. If $u = e_i$ and $v = e_j$, then the coefficient function $\pi_{u,v}$ defined in (2.3) becomes $\pi_{u,v}(x) = \langle\, \pi(x)e_j, e_i \,\rangle$, the $(i, j)^{\text{th}}$ entry of the matrix $\pi(x)$. To simplify notation, we will use $\pi_{i,j}$ to denote $\pi_{e_i,e_j}$. We now give a second important consequence of Schur's lemma regarding compact groups. The following result will be used in Chapter 4.

THEOREM 2.1.5 (Schur's Orthogonality Relations, [41, Theorem 5.8]). *Let $G$ be a compact group. For each $\pi \in \widehat{G}$, define $\mathcal{E}_\pi = \operatorname{span}\{\, \pi_{i,j} \;\big|\; i = 1, \ldots, d_\pi, j = 1, \ldots, d_\pi \,\}$. Let $\pi$ and $\pi'$ be irreducible unitary representations of $G$, and consider $\mathcal{E}_\pi$ and $\mathcal{E}_{\pi'}$ as subspaces of $L^2(G)$. Then*

1. *If $[\pi] \neq [\pi']$ then $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.*
2. *The set $\left\{\, \sqrt{d_\pi}\, \pi_{i,j} \;\big|\; i, j = 1, \ldots, d_\pi \,\right\}$ is an orthonormal basis for $\mathcal{E}_\pi$ as*

$$\langle\, \pi_{i,j}, \pi_{i',j'} \,\rangle_{L^2(G)} = \int_G \pi_{i,j}(x)\, \overline{\pi_{i',j'}(x)}\, dx = \frac{1}{d_\pi} \delta_{i,i'} \delta_{j,j'}.$$

*Here, $\delta_{i,k}$ is the Kronecker delta function.*

For a family of Hilbert spaces $\{\mathcal{H}_i\}_{i\in I}$, their (Hilbert) direct sum $\ell^2 - \bigoplus_{i\in I} \mathcal{H}_i$ is the set of all $v = (v_i)_{i\in I}$ in the Cartesian product $\prod_{i\in I} \mathcal{H}_i$ such that $\sum_{i\in I} \|v_i\|^2$ is finite. Then $\ell^2 - \bigoplus_{i\in I} \mathcal{H}_i$ is again a Hilbert space with respect to the inner product

$$\langle\, f, g \,\rangle = \sum_{i\in I} \langle\, f_i, g_i \,\rangle_{\mathcal{H}_i},$$

and the terms $\mathcal{H}_i$ are embedded in this Hilbert space as mutually orthogonal subspaces. Conversely, if $\{\mathcal{H}_i\}_{i\in I}$ are mutually orthogonal subspaces with dense linear span in a

Hilbert space $\mathcal{H}$, then we can think of $\mathcal{H}$ as the direct sum $\ell^2 - \bigoplus_{i\in I}\mathcal{H}_i$. As such, when we discuss direct sums of subspaces in a Hilbert space, we will always assume the subspaces are mutually orthogonal.

Clearly each subspace $\mathcal{E}_\pi$ defined in Theorem 2.1.5 is a closed and $R$-invariant (or $L$-invariant) subspace of $L^2(G)$. This observation yields a useful decomposition of $L^2(G)$ in terms of the irreducible representations of $G$.

THEOREM 2.1.6 (Peter-Weyl Theorem, [41, Theorem 5.12]). *Let $G$ be a compact group and let $\mathcal{E}$ be the linear span of $\bigcup_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$. Then*

1. *$\mathcal{E}$ is uniformly dense in $C(G)$;*
2. *$L^2(G) = \ell^2 - \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$; and*
3. *the set of functions*

$$\bigcup_{[\pi]\in\widehat{G}} \left\{ \sqrt{d_\pi}\pi_{i,j} \;\middle|\; i,j = 1,\ldots,d_\pi \right\}$$

*is an orthonormal (Hilbert) basis for $L^2(G)$.*

*Additionally, for $i = 1,\ldots,d_\pi$ the subspace of $\mathcal{E}_\pi$ defined by*

$$\mathcal{E}_{\pi,i} := \operatorname{span}\left\{ \pi_{i,j} \;\middle|\; 1 \le j \le d_\pi \right\}$$

*is invariant under the right regular representation $R$. The function $R$ restricted to this subspace is equivalent to the representation $\pi$. Consequently, $R$ (equivalently $L$) decomposes into a direct sum of all irreducible representations of $G$. Moreover, each $[\pi] \in \widehat{G}$ occurs in the right (or left) regular representations of $G$ with multiplicity $d_\pi$.*

In Chapter 4, we will use the Peter-Weyl theorem to obtain an orthonormal basis of eigenvectors for certain Cayley graphs.

### 2.1.1 Induced Representations

Given a locally compact group $G$ with a closed subgroup $H$, induced representation theory gives us a way to build unitary representations for $G$ from unitary representations of $H$. Let $\sigma : H \to \mathcal{U}(\mathcal{H}_\sigma)$ be a unitary representation of $H$. We denote the norm and the inner product on $\mathcal{H}_\sigma$ by $\|\cdot\|_\sigma$ and $\langle \cdot, \cdot \rangle_\sigma$, respectively. Let $C(G, \mathcal{H}_\sigma)$ denote the space of continuous functions from $G$ to $\mathcal{H}_\sigma$. Then we can consider the following space of vector-valued functions

$$\mathcal{F}_0 = \left\{ f \in C(G, \mathcal{H}_\sigma) \;\middle|\; \begin{array}{l} q(\operatorname{supp} f) \text{ is compact and} \\ f(x\xi) = \sigma(\xi^{-1})(f(x)) \text{ for } x \in G, \xi \in H \end{array} \right\},$$

where $q : G \to G/H$ is the canonical quotient map. If $G/H$ admits a $G$-invariant measure $\mu$,

$$\langle f, g \rangle_{\mathcal{F}_0} := \int_{G/H} \langle f(x), g(x) \rangle_\sigma \, d\mu(xH)$$

defines an inner product on $\mathcal{F}_0$. Let $\mathcal{F}$ denote the completion of $\mathcal{F}_0$ with respect to this inner product. Then for each $x \in G$, the left-shift operator $L(x)$ on $\mathcal{F}_0$, defined in (2.4), extends unitarily to $\mathcal{F}$, which we denote again by $L(x)$. The map defined by $L : G \to \mathcal{U}(\mathcal{F})$ is WOT-continuous and defines a unitary representation of $G$. We call this the *representation induced by* $\sigma$, and we denote it by $\operatorname{ind}_H^G(\sigma)$. In this thesis, we use induced representation machinery for semidirect product groups.

### 2.1.2 Semidirect Product of Locally Compact Groups

Let $N$ and $H$ be locally compact groups with identities $e_N$ and $e_H$, respectively. By $\operatorname{Aut}(N)$ we denote the group of automorphisms of $N$, *i.e.*, the set of all topological group isomorphisms of $N$ to itself with composition as the group product. Let the map $\alpha : H \to \operatorname{Aut}(N)$ be a group homomorphism such that

$$\psi_\alpha : N \times H \to N, \qquad (n, h) \mapsto \alpha(h)(n)$$

is continuous. Define the locally compact group $N \rtimes_\alpha H$ to be the Cartesian product $N \times H$ equipped with product topology and group actions defined as

$$(n, h) \cdot (n', h') = (n\alpha(h)(n'), hh'), \tag{2.5}$$

and

$$(n, h)^{-1} = (\alpha(h^{-1})(n^{-1}), h^{-1}).$$

It is straightforward to verify that $(e_N, e_H)$ is the identity element of $N \rtimes_\alpha H$ and that the group operations of $N \rtimes_\alpha H$ are continuous with respect to the product topology. The locally compact group $N \rtimes_\alpha H$ is called the *semidirect product* of $N$ and $H$ over $\alpha$.

Example 2.1.7 (The Affine Group). For $n \in \mathbb{N}$, let $\mathrm{M}_n(\mathbb{R})$ denote the set of $n \times n$ real matrices. Equipped with matrix addition and the topology of $\mathbb{R}^{n^2}$, the set $\mathrm{M}_n(\mathbb{R})$ can be viewed as a locally compact abelian group. Let $\mathrm{GL}_n(\mathbb{R})$ denote the subset of $\mathrm{M}_n(\mathbb{R})$ containing all $n \times n$ real matrices with nonzero determinant. Clearly $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, as the determinant is a continuous function (in fact a polynomial) in the matrix entries. So, $\mathrm{GL}_n(\mathbb{R})$ turns into a locally compact group, when equipped with matrix multiplication and the induced topology of $\mathbb{R}^{n^2}$. Elements of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{R})$ can be combined to form *affine transformations* as defined below.

Definition 2.1.8. For $x \in \mathrm{M}_n(\mathbb{R})$ and $h \in \mathrm{GL}_n(\mathbb{R})$, let $[x, h]$ denote the affine transformation of $\mathrm{M}_n(\mathbb{R})$ given by

$$[x, h]y = hy + x, \qquad \text{for } y \in \mathrm{M}_n(\mathbb{R}).$$

Let $\mathrm{M}_n(\mathbb{R}) \rtimes_\alpha \mathrm{GL}_n(\mathbb{R}) = \left\{ [x, h] \;\middle|\; x \in \mathrm{M}_n(\mathbb{R}),\, h \in \mathrm{GL}_n(\mathbb{R}) \right\}$ denote the collection of all affine transformations defined above, where $\alpha(h)(x) = h \cdot x = hx$ is matrix multiplication. Composition of transformations can be seen as the following product operation.

$$[x_1, h_1][x_2, h_2] = [x_1 + h_1x_2, h_1h_2]. \tag{2.6}$$

Then $G_n := \mathrm{M}_n(\mathbb{R}) \rtimes_\alpha \mathrm{GL}_n(\mathbb{R})$, together with product (2.6), forms a non-abelian locally compact group when given the product topology. Let $0_n$ denote the $n \times n$ zero matrix and $I_n$ denote the $n \times n$ identity matrix. It is very easy to check that $[0_n, I_n]$ is the identity of $G_n$, and $[x,h]^{-1} = [-h^{-1}x, h^{-1}]$ for $[x,h] \in G_n$.

### 2.1.3 Induced Representations for Semidirect Products

Let $G = N \rtimes_\alpha H$ be a locally compact group where $\alpha : H \to \mathrm{Aut}(N)$ defines an action of $H$ on $N$. Define $\widetilde{N} = \{ (n, e_H) \mid n \in N \}$ and $\widetilde{H} = \{ (e_N, h) \mid h \in H \}$. Then $\widetilde{N}$ and $\widetilde{H}$ are closed subgroups of $N \rtimes_\alpha H$. Clearly $\widetilde{N}$ is normal as

$$(n,h)(n', e_H)(n,h)^{-1} = \big(n\ (h \cdot n')\ (h^{-1} \cdot n^{-1}), e_H\big) \in \widetilde{N}.$$

Additionally,

1. $\widetilde{N} \cap \widetilde{H} = \{e_G\}$ and
2. $\widetilde{N}\widetilde{H} = N \rtimes_\alpha H = G$.

When these two conditions hold, we say that $\widetilde{H}$ has a *complementary subgroup* $\widetilde{N}$, and clearly every element of $G$ can be written uniquely as $\widetilde{n}\widetilde{h}$ with $\widetilde{n} \in \widetilde{N}$ and $\widetilde{h} \in \widetilde{H}$ for the product operation defined in (2.5). With the identification above in mind, we will always assume $N$ and $H$ are closed subgroups of $G$. This also yields a natural correspondence between $\widetilde{N}$ and $G/\widetilde{H}$ via the map $\widetilde{n} \mapsto \widetilde{n}\widetilde{H}$ or between $\widetilde{H}$ and $G/\widetilde{N}$ via the map $\widetilde{h} \mapsto \widetilde{h}\widetilde{N}$.

When finding an induced representation for a semidirect product group, one can induce from either subgroup $N$ or $H$, and we describe both processes below.

#### 2.1.3.1 Induction from $N$

Suppose $N$ is abelian and let $G = N \rtimes_\alpha H$ be a locally compact group where $\alpha : H \to \mathrm{Aut}(N)$ defines an action of $H$ on $N$. We denote $\alpha(h)(n) = h \cdot n$. Note

$$(e_N, h)(n, e_H)(e_N, h)^{-1} = (h \cdot n, e_H),$$

so the action of $H$ on $N$ models as conjugation on $\widetilde{N}$. The conjugation action of $G$ on $N$ induces an action of $G$ on the dual group $\widehat{N}$ given by

$$x.\nu(n) = \nu(x^{-1}.n) = \nu(x^{-1}nx), \qquad \text{for } x \in G, \nu \in \widehat{N}, n \in N.$$

For $\nu \in \widehat{N}$, let $G_\nu$ and $O_\nu$ denote the stabilizer and orbit of $\nu$ respectively, that is

$$G_\nu = \Big\{ x \in G \;\Big|\; x.\nu = \nu \Big\} \text{ and } O_\nu = \Big\{ x.v \;\Big|\; x \in G \Big\}.$$

We say $G$ acts *regularly* on $\widehat{N}$ if the following two conditions hold:

(R1) There exists a countable family $\{E_i\}_{i\in\mathbb{N}}$ of Borel sets in $\widehat{N}$ which are $G$-invariant and for each $\nu \in \widehat{N}$, we have $O_\nu = \bigcap\limits_{O_\nu \subseteq E_j} E_j$.

(R2) For each $\nu \in \widehat{N}$, the natural map $G/G_\nu \to O_\nu$ defined as $xG_\nu \mapsto x.\nu$ forms a homeomorphism.

When $G$ is $\sigma$-compact, (R2) is equivalent to

(R2') Each orbit of $\widehat{N}$ is relatively open in its closure.

For each $\nu \in \widehat{N}$, define the *little group* $H_\nu$ to be $H_\nu = G_\nu \cap H$. Clearly, as $N$ is abelian, $N \subseteq G_\nu$ and so $G_\nu = N \rtimes H_\nu$. The irreducible representations of the stabilizer $G_\nu$ are related to the little group by the following proposition.

Proposition 2.1.9 ([41, Proposition 6.41]). *If $\nu \in \widehat{N}$ and $\rho$ is an irreducible representation of $H_\nu$, then we obtain an irreducible representation of $G_\nu$, which we denote $\nu\rho$ by setting*

$$(\nu\rho)(nh) = \nu(n)\rho(h).$$

*Every irreducible representation $\sigma$ of $G_\nu$ such that $\sigma(n) = \nu(n)I$ (for $n \in N$) is of this form. Moreover, since $(\nu\rho)|_{H_\nu} = \rho$, $\nu\rho$ is equivalent to $\nu\rho'$ if and only if $\rho$ is equivalent to $\rho'$.*

*Additionally, as $N$ acts trivially on $\widehat{N}$, the $G$-orbit of $\nu \in \widehat{N}$ is the same as its $H$-orbit, and if $\nu' = x.\nu$ ($x \in H$) belongs to this orbit, the little group of $\nu$ and $\nu'$ are related by $H_{\nu'} = xH_\nu x^{-1}$. In particular, they are isomorphic.*

Remarkably, the irreducible representations of $G = N \rtimes_\alpha H$ can be completely classified in terms of the character $\nu \in \widehat{N}$ and the irreducible representations of their little groups $H_\nu$, as described in the following theorem.

THEOREM 2.1.10 ([41, Theorem 6.42]). *Suppose $G = N \rtimes_\alpha H$, where $N$ is abelian and $G$ acts regularly on $\widehat{N}$. If $\nu \in \widehat{N}$ and $\rho$ is an irreducible representation of $H_\nu$, then $\operatorname{ind}_{G_\nu}^G(\nu\rho)$ is an irreducible representation of $G$, and every irreducible representation of $G$ is equivalent to one of this form. Moreover, $\operatorname{ind}_{G_\nu}^G(\nu\rho)$ and $\operatorname{ind}_{G_{\nu'}}^G(\nu'\rho')$ are equivalent if and only if $\nu$ and $\nu'$ belong to the same orbit, say $\nu' = x\nu$, and $h \to \rho(h)$ and $h \to \rho'(x^{-1}hx)$ are equivalent representations of $H_\nu$.*

In the setting of Theorem 2.1.10, $G$ acts on $N$ by conjugation. Suppose that $H$ is a closed subgroup of $G$ such that $G = N \rtimes_\alpha H$, where $\alpha : H \to \operatorname{Aut}(N)$ is defined as $\alpha(h)(n) = hnh^{-1}$. The conjugation action of $G$ on $N$ induces an action of $G$ on the dual group $\widehat{N}$ via $\langle n, x \cdot \nu \rangle = \langle x^{-1}nx, \nu \rangle$ for $n \in N$, $x \in G$, and $\nu \in \widehat{N}$. Then the induced representation $\operatorname{ind}_N^G \nu$ has an alternate realization as follows: For $x = (n, h) \in G$, $n, k \in N, h \in H$, and $f \in L^2(H, \mathcal{H}_\pi) := \left\{ g : H \to \mathcal{H}_\pi \;\middle|\; \int_H \|g(x)\|^2 \, dx < \infty \right\}$ we have

$$\operatorname{ind}_N^G(n, h)f(k) = \pi(k^{-1}nk)f(h^{-1}k),$$

where $dx$ is the left Haar measure on $H$. The details for this realization can be found at the end of Section 2.4 in [78].

#### 2.1.3.2 Induction From $H$

The following discussion is a summary of Section 2.4, starting from Proposition 2.28, from Kaniuth and Taylor's book [78]. Only the minimal details necessary for this thesis are included, as the technicalities are beyond the scope of the current work.

Let $\pi$ be an irreducible representation of $H$. If $N$ is a complementary subgroup for $H$, we can realize the induced representation $\operatorname{ind}_H^G \pi$ on the Hilbert space

$$L^2(N, \mathcal{H}_\pi) := \left\{ g : N \to \mathcal{H}_\pi \;\middle|\; \int_N \|g(x)\|^2 \, dx < \infty \right\},$$

where $dx$ is the left Haar measure on $N$. Then for $f \in L^2(N, \mathcal{H}_\pi)$, we can explicitly write $\operatorname{ind}_H^G \pi$ in terms of the modular functions on $G$ and $H$ and the representation $\pi$ by

$$\big[(\operatorname{ind}_H^G \pi)(z)f\big](\ell) = \left[\frac{\Delta_G(h)}{\Delta_H(h)}\right]^{1/2} \pi(h)\, f(z^{-1}.\ell),$$

where $z \in G$, $\ell \in N$ and we write $z^{-1}\ell = xh^{-1}$ for some $x \in N$ and $h \in H$.

To illustrate this idea, we compute an induced representation for the affine group in the following section.

#### 2.1.3.3 Induced Representation of the Affine Group

Recall the affine group $G_n$ from Example 2.1.7 defined as

$$G_n = N \rtimes_\alpha H = \mathrm{M}_n(\mathbb{R}) \rtimes_\alpha \mathrm{GL}_n(\mathbb{R}),$$

with $\alpha(h)x = hx$.

**Haar Integration.** We now specifically describe the Haar measure for $G_n$. In what follows, all the functions appearing in the integration formulas are integrable and defined on the appropriate domains. First, we equip $\mathrm{M}_n(\mathbb{R})$ with Lebesgue measure under the identification with $\mathbb{R}^{n^2}$, and let $\int_{\mathrm{M}_n(\mathbb{R})} f(x)\, dx$ denote Lebesgue integration. That is,

$$dx = dx_{1,1}dx_{1,2}\cdots dx_{1,n}dx_{2,1}\cdots dx_{n,1}\cdots dx_{n,n} \quad \text{if } x = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{pmatrix}. \tag{2.7}$$

Proposition 2.1.11. *For* $\mathrm{GL}_n(\mathbb{R})$, *the left Haar integration is given by*

$$\int_{\mathrm{GL}_n(\mathbb{R})} g(h)\, \frac{dh}{|\det h|^n},$$

*where* $dh$ *is the Lebesgue measure in* (2.7) *and* $h = (h_{i,j})_{i,j=1}^n$ *is a generic element of* $\mathrm{GL}_n(\mathbb{R})$. *This measure is also right invariant, i.e.,* $\mathrm{GL}_n(\mathbb{R})$ *is unimodular.*

*Proof.* Let $a, h \in \mathrm{GL}_n(\mathbb{R})$. Denote the change of variables map $h \mapsto ah$ by $A$. We denote the Jacobian matrix of this transform by $\mathrm{J}_A$. Then by indexing the rows and columns of $\mathrm{J}_A$ by $(1,1), (2,1), \ldots, (n,1), (1,2), \ldots, (n,2), \ldots, (n,n)$, it is easy to see that $\mathrm{J}_A$ is block diagonal with $n$ copies of $a$ on the diagonal and 0 elsewhere. So $|\,\mathrm{J}_A\,| = |\det a|^n$. Therefore

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} f(ah) \frac{dh}{|\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} f(h) \frac{d(a^{-1}h)}{|\det a^{-1}h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} f(h) \frac{|\det a^{-1}|^n\, dh}{|\det a^{-1}|^n |\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} f(h) \frac{dh}{|\det h|^n}.$$

A similar calculation shows that the measure is right invariant and so $\mathrm{GL}_n(\mathbb{R})$ is unimodular. $\square$

Proposition 2.1.11 shows that, for any $h' \in \mathrm{GL}_n(\mathbb{R})$ and compactly supported function $g : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$,

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} g(h'h)\, \frac{dh}{|\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} g(hh')\, \frac{dh}{|\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} g(h^{-1})\, \frac{dh}{|\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} g(h)\, \frac{dh}{|\det h|^n}.$$

We now describe left Haar integration on $G_n$. For a compactly supported function $f : G_n \to \mathbb{C}$,

$$\int\limits_{G_n} f\big([x,h]\big)\, d[x,h] = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} f\big([x,h]\big)\, \frac{dx\, dh}{|\det(h)|^{2n}}. \tag{2.8}$$

Proposition 2.1.12. *The integration defined in* (2.8) *is invariant under left translations.*

*Proof.* Considering that

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} f\big([y,k][x,h]\big)\, \frac{dx\, dh}{|\det(h)|^{2n}} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} f\big([y+kx, kh]\big)\, \frac{dx\, dh}{|\det(h)|^{2n}},$$

the change of variable maps are given by $x \mapsto k^{-1}(-y+x)$ and $h \mapsto k^{-1}h$. Then the right hand side becomes

$$\int_{\mathrm{GL}_n(\mathbb{R})} \int_{\mathrm{M}_n(\mathbb{R})} f\big([x,h]\big) \frac{d\big(k^{-1}(-y+x)\big)\, d(k^{-1}h)}{|\det(k^{-1}h)|^{2n}} = \int_{\mathrm{GL}_n(\mathbb{R})} \int_{\mathrm{M}_n(\mathbb{R})} f\big([x,h]\big) \frac{|\det k^{-1}|^{2n}\, dx\, dh}{|\det k^{-1}|^{2n} |\det h|^{2n}}.$$

Therefore the integration defined in (2.8) is left invariant. □

However, this integration is not right invariant, as the change of variable maps in this case are $x \mapsto (x - hy)$ and $h \mapsto hk^{-1}$. By a similar computation as in the proof of Proposition 2.1.12, we have the following formula for handling the case of right translation.

$$\int_{G_n} f\big([x,h][y,k]\big)\, d[x,h] = |\det(k)|^n \int_{G_n} f\big([x,h]\big)\, d[x,h],$$

for every $[y,k] \in G_n$, showing that the modular function from (2.1) for the affine group is $\Delta_{G_n}([y,k]) = |\det k|^{-n}$.

**Induced Representation of $G_n$.** Let $\tau : \mathrm{GL}_n(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\tau) \cong \mathbb{C}$ be the trivial representation of $\mathrm{GL}_n(\mathbb{R})$, *i.e.*, $\tau(h) = 1$ for every $h \in \mathrm{GL}_n(\mathbb{R})$.

Then for $z = [x,h] \in G_n$ and $\ell = [y, I_n] \in \widetilde{\mathrm{M}}_n(\mathbb{R})$, we have

$$z^{-1}\ell = [-h^{-1}x, h^{-1}]\,[y, I_n] = [-h^{-1}(x-y), h^{-1}] = [h^{-1}(y-x), I_n]\,[0_n, h]^{-1}.$$

By identifying $[w, I_n] \in G_n$ with $w \in \mathrm{M}_n(\mathbb{R})$, we have for $f \in L^2(\mathrm{M}_n(\mathbb{R}))$:

$$\begin{aligned}\left[\Big(\mathrm{ind}_{\mathrm{GL}_n(\mathbb{R})}^{G_n} \tau\Big)[x,h] f\right](y) &= \left[\frac{\Delta_{G_n}(h)}{\Delta_{\mathrm{GL}_n(\mathbb{R})}(h)}\right]^{1/2} \tau(h)\, f\big([x,h]^{-1}y\big) \\ &= |\det h|^{-n/2} f\big(h^{-1}(y-x)\big),\end{aligned} \tag{2.9}$$

where the modular functions for $\mathrm{GL}_n(\mathbb{R})$ and $G_n$ were determined in Propositions 2.1.11 and 2.1.12, respectively.

The above machinery for determining representations for semidirect product groups from representations of closed subgroups will be incredibly useful during Chapter 5. In particular, we will make extensive use of the representation given in (2.9).

## 2.2 Fourier Analysis on LCA Groups

In the current section, we introduce the *Fourier transform* in the context of locally compact abelian (LCA) groups. For the remainder of this section, assume $G$ is a LCA group. This integral transform provides a representation of a function $f \in L^1(G)$ in terms of the irreducible unitary representations of $G$. This is accomplished through the map $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$, given by

$$\mathcal{F}f(\xi) = \int_G f(x)\, \overline{\xi(x)}\, dx.$$

We denote $\mathcal{F}f(\xi)$ by $\widehat{f}(\xi)$. If $G$ is a locally compact abelian group, then $\widehat{G}$ is also a locally compact abelian group [93, Theorem 1.2.6]. Consequently, $\widehat{G}$ also has a Haar measure. Once a Haar measure for $G$ and identification of $\widehat{G}$ has been chosen, and the Haar measure on $\widehat{G}$ has been scaled appropriately, we have the following result.

THEOREM 2.2.1 (Fourier Inversion Theorem [41, Theorem 4.33]). *If $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} \widehat{f}(\xi)\, \xi(x)\, d\xi, \qquad \text{for a.e. } x \in G,$$

*where $d\xi$ is the Haar measure of $\widehat{G}$. If $f$ is continuous, this relation holds for every $x \in G$.*

Most often, $G$ is thought of as the time-domain of a signal $f \in L^1(G)$, and $\widehat{G}$ is thought of as the frequency domain—notions stemming from the applications of this theory to traditional audio signals as functions on the real line.

In the current work, we primarily consider $L^2(G)$, the space of square-integrable functions with respect to Haar measure. In this case, the Fourier transform extends to a unitary operator by the following classical result.

THEOREM 2.2.2 (The Plancherel Theorem [93, Theorem 1.6.1]). *The Fourier transform, restricted to $(L^1 \cap L^2)(G)$, is an isometry (with respect to the $L^2$-norms) onto a dense linear subspace of $L^2(\widehat{G})$. Hence it may be extended, in a unique manner, to a unitary isomorphism $\mathcal{P} : L^2(G) \to L^2(\widehat{G})$.*

REMARK 2.2.3. As is standard in the literature, we will use $\widehat{f}$ to denote both the Fourier transform and Plancherel transform $\mathcal{P}f$ of a function $f$, as it will be clear from context which one is being used.

We now provide concrete examples of the objects defined above with respect to the groups which will be discussed in future chapters.

### 2.2.1 Finite Cyclic Groups

Let $G = \mathbb{Z}_N$. To put the discussion in the context of harmonic analysis, we think of $\mathbb{Z}_N$ as a compact group equipped with the discrete topology. As it is customary for finite groups, we equip $\mathbb{Z}_N$ with the normalized counting measure $\mu$, *i.e.*, $\mu(E) = \frac{|E|}{N}$ for every subset $E$ of $\mathbb{Z}_N$ with $|E|$ distinct elements. The dual object $\widehat{\mathbb{Z}_N}$ is the group of characters $\chi_k$ with $k \in \{0, \dots, N-1\}$, where each $\chi_k : \mathbb{Z}_N \to \mathcal{U}(\mathbb{C})$ is defined by

$$\chi_k(m) = e^{\frac{2\pi ikm}{N}}, \text{ for } m \in \mathbb{Z}_N = \{0, 1, \dots, N-1\}. \tag{2.10}$$

Clearly, $\{\chi_k \mid k = 0, \dots, N-1\}$ forms an orthonormal basis of $L^2(\mathbb{Z}_N, \mu)$. The classical Fourier transform of a function $f \in L^1(\mathbb{Z}_N)$ at $n \in \mathbb{Z}_N$ is defined by

$$\widehat{f}(n) = \int_G f(m)\overline{\chi_n(m)}\, d\mu(m) = \frac{1}{N}\sum_{m=0}^{N-1} f(m)\overline{\chi_n(m)}. \tag{2.11}$$

The inverse Fourier transform can be written in a similar manner, once an appropriate Haar measure is fixed for $\widehat{\mathbb{Z}_N}$. Guided by the theory of commutative harmonic analysis, we think of $\widehat{\mathbb{Z}_N}$ as a discrete group (as it is the dual of a compact group [41, Proposition 4.5]), and we equip it with the usual counting measure. The inverse Fourier transform then becomes

$$f(m) = \int_{\widehat{G}} \widehat{f}(n)\chi_n(m)\, dn = \sum_{n=0}^{N-1} \widehat{f}(n)\chi_n(m), \quad \text{for } m \in \mathbb{Z}_N, \tag{2.12}$$

where $dn$ is the counting (Haar) measure on $\widehat{G}$.

### 2.2.2 The Group $\mathrm{M}_n(\mathbb{R})$

In Chapter 5, we work with the Fourier transform on $L^2(\mathrm{M}_n(\mathbb{R})) \cong L^2(\mathbb{R}^{n^2})$, where we identify the LCA group $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ by the map

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix} \overset{\Phi}{\longmapsto} \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & \dots & x_{n,1} & \dots & x_{n,n} \end{pmatrix}^\top .$$

Recall that $M^\top$ denotes the transpose of the matrix $M$. Similar to the previous section, for every $k \in \mathrm{M}_n(\mathbb{R})$ we define characters $\chi_k : \mathrm{M}_n(\mathbb{R}) \to \mathcal{U}(\mathbb{C})$ by

$$\chi_k(y) = e^{2\pi i \,\mathrm{Tr}(ky)},$$

where $\mathrm{Tr}(x)$ is the trace of the matrix $x$. This leads to the Fourier transform for $f \in L^1(\mathrm{M}_n(\mathbb{R}))$ defined as

$$\mathcal{F}f(\chi_k) = \widehat{f}(\chi_k) = \int\limits_{\mathrm{M}_n(\mathbb{R})} f(x)\overline{\chi_k(x)}\, dx = \int\limits_{\mathrm{M}_n(\mathbb{R})} f(x) e^{-2\pi i \,\mathrm{Tr}(kx)}\, dx.$$

Then the inverse Fourier transform for $\widehat{f} \in L^1(\widehat{\mathrm{M}_n(\mathbb{R})})$ is given by

$$\mathcal{F}^{-1}\widehat{f}(x) = \int\limits_{\mathrm{M}_n(\mathbb{R})} \widehat{f}(\chi_k)\chi_k(x)\, dk = \int\limits_{\mathrm{M}_n(\mathbb{R})} \widehat{f}(\chi_k) e^{2\pi i \,\mathrm{Tr}(kx)}\, dk.$$

Note, in both cases, integration over $\mathrm{M}_n(\mathbb{R})$ is with respect to the Lebesgue integral

$$\int\limits_{\mathrm{M}_n(\mathbb{R})} f(x)\, dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_{1,1}, \dots, x_{n,n})\, dx_{11} \cdots dx_{n,n},$$

$$\int\limits_{\mathrm{M}_n(\mathbb{R})} \widehat{f}(\chi_k)\, dk = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \widehat{f}(\chi_{k_{1,1},\dots,k_{n,n}})\, dk_{1,1} \cdots dk_{n,n}.$$

We now define several operators which will prove useful in our discussions of frames in later chapters.

Definition 2.2.4. Consider the following families of unitary operators on $L^2(\mathrm{M}_n(\mathbb{R}))$:

1. For $a \in \mathrm{M}_n(\mathbb{R})$, the translation operator $T_a$ is defined by

$$T_a : L^2(\mathrm{M}_n(\mathbb{R})) \to L^2(\mathrm{M}_n(\mathbb{R})), \qquad (T_a f)(x) := f(-a + x), x \in \mathrm{M}_n(\mathbb{R}).$$

2. For $\xi \in \mathrm{M}_n(\mathbb{R})$, the modulation operator $M_\xi$ is defined by

$$M_\xi : L^2(\mathrm{M}_n(\mathbb{R})) \to L^2(\mathrm{M}_n(\mathbb{R})), \qquad (M_\xi f)(x) := \chi_\xi(x) f(x), x \in \mathrm{M}_n(\mathbb{R}).$$

3. For $h \in \mathrm{GL}_n(\mathbb{R})$, the dilation operator $D_a$ is defined by

$$D_h : L^2(\mathrm{M}_n(\mathbb{R})) \to L^2(\mathrm{M}_n(\mathbb{R})), \qquad (D_h f)(x) := \frac{1}{\sqrt{|\det h|^n}} f(h^{-1}x), x \in \mathrm{M}_n(\mathbb{R}).$$

Remark 2.2.5. The operator $T_a$ is identical to $L(a)$, where $L$ is the left regular representation on $G = \mathrm{M}_n(\mathbb{R})$. The operator $M_\xi$ is pointwise multiplication with the character $\chi_\xi$, with characters often being thought of as the 'pure states' or frequencies of a signal $f \in L^2(\mathrm{M}_n(\mathbb{R}))$. The operator $D_h$ results in scaling by the matrix $h$, where the determinant factor is required for $D_h$ to be a unitary operator.

As it will prove useful in discussions of wavelet and Gabor systems in future chapters, we now compute the Fourier transform for each of these families of operators.

Proposition 2.2.6. *Let $a, k, \xi \in \mathrm{M}_n(\mathbb{R})$ and let $h \in \mathrm{GL}_n(\mathbb{R})$. For any $f \in L^2(\mathrm{M}_n(\mathbb{R}))$, we have*

1. $(\widehat{T_a f})(\chi_k) = \overline{\chi_k(a)}\widehat{f}(\chi_k)$.
2. $(\widehat{D_h f})(\chi_k) = |\det h|^{n/2} f(\chi_{kh})$.
3. $(\widehat{M_\xi f})(\chi_k) = \widehat{f}(\chi_{-\xi+k}) = T_\xi \widehat{f}(\chi_k)$.

*Proof.* We prove the statement for the dilation operator, as the other two are similar. Let $f \in (L^1 \cap L^2)(\mathrm{M}_n(\mathbb{R}))$, then we have

$$\begin{aligned}
\widehat{D_h f}(\chi_k) = \int_{\mathrm{M}_n(\mathbb{R})} D_h f(x)\overline{\chi_k(x)}\, dx &= |\det h|^{-n/2} \int_{\mathrm{M}_n(\mathbb{R})} f(h^{-1}x)\overline{\chi_k(x)}\, dx \\
&= |\det h|^{n/2} \int_{\mathrm{M}_n(\mathbb{R})} f(x)\overline{\chi_k(hx)}\, dx \\
&= |\det h|^{n/2} \int_{\mathrm{M}_n(\mathbb{R})} f(x)\overline{\chi_{kh}(x)}\, dx \\
&= |\det h|^{n/2}\, \widehat{f}(\chi_{kh}).
\end{aligned}$$

The extension to general $f \in L^2(\mathrm{M}_n(\mathbb{R}))$ follows from a standard density argument. □

## Chapter 3

## BACKGROUND AND NOTATIONS: DISCRETE FRAME THEORY

This chapter provides historical context and a survey of the most important results from frame theory that are required in this thesis. Formally, a discrete frame is defined as:

Definition 3.0.1. Let $X$ be a countable set. A set of vectors $\left\{\phi_i\right\}_{x\in X}$ is a *discrete frame* for a Hilbert space $\mathcal{H}$ if there exist positive constants $A$ and $B$ which depend exclusively on $\mathcal{H}$ and the set of function $\left\{\phi_i\right\}_{x\in X}$ such that

$$A\left\|f\right\|_{\mathcal{H}}^2 \leq \sum_{x\in X}\left|\langle f,\phi_i\rangle\right|^2 \leq B\left\|f\right\|_{\mathcal{H}}^2 \tag{3.1}$$

for every function $f\in\mathcal{H}$.

Without loss of generality, we will always assume our index set $X$ is given by $\mathbb{N}$ throughout this chapter.

This chapter is organized as follows. In Section 3.1, we give a brief historical perspective on the development of discrete frame theory. In Section 3.2, we introduce notations and provide fundamental tools used in the subsequent chapters. First, we review the general theory of discrete frames, define the frame operator, and describe some of its properties. We then provide a concrete example of a frame for $\mathbb{R}^2$, and also discuss several examples demonstrating the difficulties of constructing frames in infinite-dimensional spaces. We conclude with a discussion of 'stability' and what that means for frame systems. For a complete introduction to frame theory, we refer to the book of Ole Christensen [21].

## 3.1 Historical Perspective

Discrete frames were initially introduced in 1952 by Duffin and Schaeffer in [34], who were interested in studying non-harmonic expansions of entire functions of exponential type. We include their original definition, which restricted the functions under consideration.

In [34], a set of functions $\left\{e^{i\lambda_n t}\right\}_{n=1}^{\infty}$ is a *frame* over an interval $(-\gamma, \gamma)$ if there exist positive constants $A$ and $B$, which depend exclusively on $\gamma$ and the set of functions $\left\{e^{i\lambda_n t}\right\}$, such that

$$A \le \frac{\frac{1}{2\pi}\sum_{n=1}^{\infty}\left|\int_{-\gamma}^{\gamma} g(t)e^{i\lambda_n t}\,dt\right|^2}{\int_{-\gamma}^{\gamma}|g(t)|^2\,dt} \le B$$

for every function $g \in L^2\left[(-\gamma, \gamma)\right]$.

This is also the definition which appears in Young's 1980 textbook [114] on the subject of non-harmonic Fourier expansions, which appears to be the next major contribution in the area.

An important step in the development of discrete frame theory coincides with developments in general wavelet analysis in the 1980s. For the purposes of this introduction, we will restrict discussion to the 1-dimensional version, but we will discuss higher-dimensional generalizations in Chapter 5. In the 1-dimensional setting, we can state the primary question in wavelet analysis as: what conditions are necessary on a function $\psi \in L^2(\mathbb{R})$ so that

$$f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_f(a,b)\psi^{a,b}\,da\,db, \qquad \forall f \in L^2(\mathbb{R}), \tag{3.2}$$

where $\psi^{a,b}$ are dilations by $a$ and translations by $b$ of the original function $\psi$, and $c_f(a,b)$ are coefficients? This integral reproducing formula was first introduced by Morlet and Grossmann in 1984 [64], where they determined an admissibility condition on $\psi$ for the above formula to hold true. This transform later became known as the continuous wavelet transform (CWT).

Grossmann proposed restricting the reproducing formula to a countable number of functions $\psi^{a,b}$, which he published in a seminal paper in 1986 with Daubechies and Meyer, giving the earliest constructions of discrete versions of the CWT [32] (*i.e.*, a discrete frame). The authors introduced discretized versions of both Gabor and wavelet systems in terms of the action of an irreducible representation of the Heisenberg and affine groups, respectively, on a fixed function (wavelet). These provide 'good' examples, wherein certain complications in reconstruction, which we discuss later, do not arise. They were the first to generalize the original frame definition above to any separable Hilbert space.

This work, along with other important early papers by Daubechies [31] and Heil and Walnut [68], led to an explosion of interest in the theory of discrete frames beginning in the early 1990s and continuing to this day. In practice, recovery of signals from the frame coefficients in the setting of infinite dimensional Hilbert spaces requires iterative methods, and so one desirable property is rapid convergence of any reconstruction algorithm. The convergence rate of reconstruction is strongly related to the gap between the bounds $A$ and $B$ in (3.1). Frames are therefore explicitly constructed and optimized for specific classes of data at hand, which makes generalizing the known frame constructions to other datasets a highly non-trivial task.

For instance, the classical Fourier basis efficiently represents a harmonic signal in terms of its frequencies (global property of the signal); however, it represents localized signals (*i.e.*, functions with compact support) quite inefficiently. The study of wavelet transforms has proven a fruitful strategy for creating novel frames in recent years. Classical 1-dimensional wavelets may be formed in such a way as to have rapid decay in both spatial and frequency domains, and thus may be used to construct frames for the space of localized 1-dimensional signals. Unfortunately, generalizing classical wavelet transform constructions even to the setting of 2-dimensional signals is not often useful. The resulting wavelets are often isotropic and cannot provide information about the geometry of anisotropic signals in two dimensions. However, instances of 2-dimensional wavelet-type transforms, such as curvelet transforms (introduced in 2000, see [17, 18, 19, 107, 108]) and shearlet transforms (introduced in 2006, see [35, 65, 66, 80, 98]) have proven extremely successful in the analysis and denoising of signals presenting anisotropic features.

Our focus in this thesis will be the construction of discrete frames in two settings. First, we describe constructions of Gabor-type discrete frames (defined momentarily) on the irregular domain of the vertex set of a graph. Second, we will construct discrete frames for signals defined on the regular, Euclidean domain of $\mathrm{M}_n(\mathbb{R})$ by discretizing a higher-dimensional version of the continuous wavelet transform. We now provide a brief history of each problem.

**Gabor Frames.** In the classical setting, *the windowed Fourier transform* (WFT) is a continuous analogue of a discrete frame and was introduced in a seminal paper by Gabor in 1946 [51]. It was designed to measure the "frequency variations" of audio signals. The WFT represents these 1-dimensional signals highly redundantly as 2-dimensional time-frequency images which allows for detailed analysis of how the frequency content of a signal changes over time.

In general, this involves pointwise multiplication by a smooth, localized function, or *time window*, and modulation by a frequency. Formally, a window function $g$ is a square-integrable function with sufficiently rapid decay such that the mapping defined by $x \mapsto xg(x)$ is also square-integrable. By convention one usually takes $\|g\|_{L^2(\mathbb{R})} = 1$. Defining $g_{x,k}(t) := g(t-x)e^{2\pi ixt}$ for $x, k \in \mathbb{R}$, the windowed Fourier transform is given by

$$S\{f\}(x,k) := \langle\, f, g_{x,k}\,\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(t)\overline{g(t-x)}e^{-2\pi ikt}\, dt. \tag{3.3}$$

Gabor frames generally refer to discrete versions of the above transform. That is, by sampling a discrete subset of $\mathbb{R}^2$ to obtain a discrete frame defined through translations and modulations of the function $g$.

**Frames for Graph Signals.** Much of the recent interest in developing techniques for graph signal processing appears to stem from the introduction of the graph Fourier transform originally defined by Hammond, Vandergheynst, and Gribonval in their 2011 article [67]. Continued development in the field has primarily progressed along two distinct branches. The first approach comes from the viewpoint of algebraic signal processing, which considers general, (possibly directed) graphs [20, 94, 96]. The second approach focuses on undirected graphs using the graph Laplacian as the basic tool to generalize the signal processing toolkit, [67, 101]. For a complete historical review, we refer to the recent survey article by Ortega *et al.* [90].

Our interest will focus on the development of frames defined for graph signals. The original paper introducing the graph Fourier transform, [67], generalizes the wavelet transform as in (3.2) to this setting. The other primary method to generate frames for signals defined on graphs has been through defining the discrete analogue of (3.3), the windowed Fourier transform, to the graph setting. This is the method presented in Chapter 4.

**Continuous Frames.** Both the continuous wavelet transform given in (3.2) and the windowed Fourier transform in (3.3) are special cases of continuous frames.

Definition 3.1.1. A *continuous frame* for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{\phi_x\}_{x\in X}$, with $X$ a locally compact Hausdorff space equipped with a positive Radon measure $\mu$, satisfying

$$A\|f\|_{\mathcal{H}}^2 \leq \int_X |\langle f, \phi_x \rangle_{\mathcal{H}}|^2 \, d\mu(x) \leq B\|f\|_{\mathcal{H}}^2, \qquad \forall f \in \mathcal{H}, \tag{3.4}$$

for some positive real numbers $A$ and $B$.

Notice that when $\mu$ is the counting measure on a countable space $X$, this agrees with the previous definition of a discrete frame given in (3.1). The definition is also that of *equivalent norms* between the original Hilbert space $\mathcal{H}$ and $L^2(X)$.

The term 'continuous frame' is attributed to Ali, Antoine, and Gazeau in [2], although the concept did not originate with them. General continuous frames for $L^2(\mathbb{R}^n)$ were known to Calderón in the 1960s (see [16])—as such, some authors refer to the following as the *Calderón reproducing formula* (for example, [81]). Essentially, when $\{\phi_x\}_{x\in X}$ is a continuous frame, there exist vectors $\{\widetilde{\phi}_x\}_{x\in X}$, called the *dual frame*, such that

$$f = \int_X \langle f, \widetilde{\phi}_x \rangle \, \phi_x \, d\mu(x) \qquad \forall f \in \mathcal{H},$$

where the integral is interpreted in the weak sense. Continuous frames were also studied by Kaiser in [77, Chapter 4] under the name *generalized frames*.

**The Discretization Problem.** The interest for physicists in integral reproducing formulas, as in Equation (3.4), dates back to Schrödinger in 1926 [97], where he introduced the idea of *coherent states*, an important idea in quantum mechanics. The links between coherent states, Gabor systems, and wavelet systems are well-explained in both the original 1986 paper of Daubechies, Grossmann, and Meyer [32], as well as Daubechies' 1990 article [31]. The question of classifying which continuous reproducing formulas (as special cases of coherent states) could be discretized was first posed by the mathematical physicists Ali, Antoine, and Gazeau in 2000 in their physics textbook [3], and is known in the literature as the *discretization problem*.

Existence proofs of such discretizations for general Banach spaces date back to the late 1980s and the work of Feichtinger and Gröchenig in the articles [38, 39]. Gröchenig further developed the theory for Hilbert spaces in [63], which gives an early result stating that if the orbit of an irreducible, unitary, integrable representation is dense in the Hilbert space, then a discrete frame can be constructed [63, Theorem T]. This result was strengthened in 2007 by Führ and Gröchenig to show that if the representation is an irreducible, unitary, *square-integrable* group representation, then the continuous reproducing formula may always be sampled to obtain a discrete frame [48, Theorem 4.1]. This question was only fully answered by Freeman and Speegle in a very recent article; the authors prove that if the reproducing function (*i.e.*, the map $\psi \mapsto \psi^{a,b}$ in (3.2)) is bounded almost everywhere, then the continuous frame formula may be sampled to obtain a discrete frame [44, Theorem 1.3].

Building on several of the previously mentioned works, Führ and Oussa, [50], found large classes of Lie groups $G$ for which $L^2(G)$ admits discrete frames of translates. Additionally, they proved several necessary and some sufficient conditions for existence of such discrete frames. In contrast, it was shown by Kutyniok in [79, Proposition 4.11] that $L^2(\mathbb{R}^n)$ does not admit discrete frames of pure translates or dilations, in the sense that no collections of the form $\bigcup_{k=1}^{r}\big\{g_k(x-a)\big\}_{a\in\Gamma_k}$ or $\bigcup_{k=1}^{r}\big\{b^{-1/2}g_k(b^{-1}x)\big\}_{b\in\Gamma_k}$ can form a frame for $L^2(\mathbb{R}^n)$. This demonstrates the necessity of more complex methods, such as the one we explore in Chapter 5.

## 3.2 Discrete Frame Theory

The main question regarding discrete frames is one we have already seen asked in the context of wavelet and Gabor systems as discussed in the previous section: what conditions are necessary on an arbitrary sequence of vectors $\big\{\phi_i\big\}_{i\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ so that there exist coefficients $\big\{c_f(i)\big\}_{i\in\mathbb{N}}$ such that

$$f = \sum_{i\in\mathbb{N}} c_f(i)\phi_i$$

for all $f \in \mathcal{H}$?

When the upper frame bound holds in Definition 3.0.1, then the sequence $\big\{\phi_i\big\}_{i\in\mathbb{N}}$ is called a *Bessel sequence* in $\mathcal{H}$ with Bessel bound $B$. For Bessel sequences, we have the following result.

Theorem 3.2.1 ([21, Theorem 3.2.3]). *Let $\big\{\phi_i\big\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and $B > 0$ be given. Then $\big\{\phi_i\big\}_{i\in\mathbb{N}}$ is a Bessel sequence with Bessel bound $B$ if and only if*

$$T\{c_i\}_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}} c_i\phi_i.$$

*is a well-defined bounded operator from $\ell^2(\mathbb{N})$ into $\mathcal{H}$ and $\|T\|_{\text{op}} \le \sqrt{B}$, where the operator norm $\|T\|_{\text{op}}$ is given by*

$$\|T\|_{\text{op}} = \sup_{f\in\mathcal{H}}\Big\{\ \|Tf\|_{\mathcal{H}}\ \Big|\ \|f\|_{\mathcal{H}} = 1\ \Big\}.$$

Having a Bessel sequence is not sufficient to always obtain series expansions. However, the existence of both the upper and lower frame bounds in Definition 3.0.1 is a sufficient condition for this to occur. In that case, the bounded operator $T$ in Theorem 3.2.1 is called the *synthesis*, *pre-frame*, or *reconstruction* operator. Then by [21, Lemma 3.2.1], the adjoint of $T$, denoted $T^* : \mathcal{H} \to \ell^2(\mathbb{N})$, is given by

$$T^* f = \big\{ \langle f, \phi_i \rangle \big\}_{i=1}^{\infty}.$$

Example 3.2.2 (Mercedes-Benz Frame). One of the nicest, non-trivial examples of a frame for a finite dimensional Hilbert space is the so called Mercedes-Benz frame for $\mathbb{R}^2$. The frame $\{\phi_i\}_{i=1}^{3}$ is given by the vectors

$$\phi_1 = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}, \qquad \phi_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}.$$

In finite dimensional vector spaces, the synthesis operator is given by the matrix with the frame vectors as columns:

$$T = \left( \phi_1 \middle| \phi_2 \middle| \phi_3 \right).$$

One can easily check that, for this example, $TT^* = I_2$, where $I_n$ is the $n \times n$ identity matrix. With this observation, it is clear that for any vector $v \in \mathbb{R}^2$ we have

$$v = (TT^*)v = \sum_{i=1}^{3} \langle v, \phi_i \rangle \, \phi_i.$$

Let us take a moment to consider why this frame might provide a 'better' representation for a vector over the representation in the standard orthonormal basis of $\mathbb{R}^2$. First note that the synthesis operator maps $\mathbb{R}^3$ to $\mathbb{R}^2$, so clearly $T$ has a non-trivial kernel. In fact, $\ker(T) = \operatorname{span}\big\{(1,1,1)^\top\big\}$. Therefore, any shift of the frame coefficients by a multiple of the all ones vector will still result in perfect reconstruction. For example, if one takes the standard basis vector $e_1$, then

$$T^* e_1 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

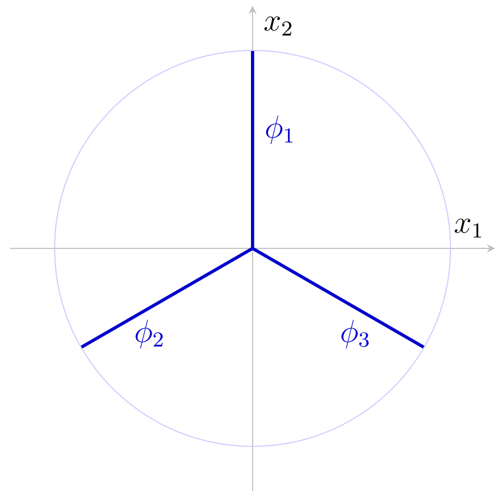



**Figure 3.1:** The Mercedes-Benz frame in $\mathbb{R}^2$.

However, if the encoded frame coefficients are all perturbed by any constant $\alpha \in \mathbb{R}$, then upon reconstruction we have

$$T \begin{pmatrix} \alpha \\ \alpha - \frac{1}{\sqrt{2}} \\ \alpha + \frac{1}{\sqrt{2}} \end{pmatrix} = T \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} + T \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So a perturbation by any vector in the kernel of the synthesis operator vanishes when recovering the vector. Since the dimension of the range of $T$ is fixed, as the level of redundancy increases, so does the dimension of the kernel. This allows for additional error cancellation to occur when reconstructing the function from its frame coefficients. As the synthesis operator for a basis will *always* have a trivial kernel, this phenomenon never happens with regards to reconstruction from a basis.

Define the operator $S := TT^* : \mathcal{H} \to \mathcal{H}$, which is usually referred to as the *frame operator*. In the previous example, $S$ was the identity operator on $\mathcal{H}$. When this occurs, the frame is called a *Parseval tight frame* or 1-tight frame, as the corresponding frame bounds are $A = B = 1$. In general, if $TT^* = kI$, then the frame bounds are $A = B = k$, and we say the columns of $T$ form a *$k$-tight frame*. Tight frames are particularly 'nice' examples of frames.

For optimal frame bounds $A$ and $B$, the *condition number* for the frame is defined as the ratio $\frac{B}{A}$. This quantity is clearly bounded below by 1. The condition number can be thought of as a measurement on how perturbations (errors) in the frame coefficients propagate in calculations involving the frame. The closer the condition number is to its lower bound of 1, the better conditioned the frame system. As a result, there is a great deal of interest in constructing tight frames like the one illustrated in Example 3.2.2. For tight frames, the reconstruction formula (provided shortly) is similar to that given for an orthonormal basis. As such, Parseval tight frames can be thought of as the generalizations of orthonormal bases in the setting of frames.

The following results provide some properties of the operator $S$ and how it can be used to generate a new family of vectors which can be useful in reconstruction formulas.

LEMMA 3.2.3 ([21, Lemma 5.1.5]). *Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a frame with frame operator $S$ and frame bounds $A, B$. Then the following hold:*

1. *$S$ is bounded, invertible, self-adjoint, and positive.*
2. *$\{S^{-1}\phi_i\}_{i\in\mathbb{N}}$ is a frame with frame bounds $B^{-1}, A^{-1}$; if $A, B$ are the optimal frame bounds for $\{\phi_i\}_{i\in\mathbb{N}}$, then the bounds $B^{-1}, A^{-1}$ are optimal for $\{S^{-1}\phi_i\}_{i\in\mathbb{N}}$. The frame operator for $\{S^{-1}\phi_i\}_{i\in\mathbb{N}}$ is $S^{-1}$.*

When $\{\phi_i\}_{i\in\mathbb{N}}$ forms a frame, but not a tight frame, reconstruction requires a second family of vectors, as described in the following definition.

DEFINITION 3.2.4. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a discrete frame for a Hilbert space $\mathcal{H}$. A collection of vectors $\{\varphi_i\}_{i\in\mathbb{N}}$ is said to be a *dual frame* to $\{\phi_i\}_{i\in\mathbb{N}}$ if

$$f = \sum_{i\in\mathbb{N}} \langle f, \varphi_i \rangle \phi_i, \qquad \forall f \in \mathcal{H}.$$

In particular, the set of vectors $\{S^{-1}\phi_i\}_{i\in\mathbb{N}}$ described in Lemma 3.2.3, known as the *canonical dual frame*, has the following properties.

THEOREM 3.2.5 ([21, Theorem 5.1.6]). *Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a frame with frame operator $S$. Then for all $f \in \mathcal{H}$, we have*

$$f = \sum_{i\in\mathbb{N}} \langle f, S^{-1}\phi_i \rangle \phi_i = \sum_{i\in\mathbb{N}} \langle f, \phi_i \rangle S^{-1}\phi_i.$$

*Both series converge unconditionally for all $f \in \mathcal{H}$.*

Combining some of our previous discussion, we have that if the vectors form a tight frame then $S = AI$. Clearly, in this case, $S^{-1} = \frac{1}{A}I$, so the reconstruction formula is given by $f = A^{-1}\sum_{i\in\mathbb{N}} \langle f, \phi_i \rangle \phi_i$. By scaling the vectors as $\{A^{-1/2}\phi_i\}_{i\in\mathbb{N}}$, any $A$-tight frame becomes a Parseval tight frame. More generally, the optimal frame bounds $A$ and $B$ are related to the operator norm of $S$ as follows.

THEOREM 3.2.6 ([21, Theorem 5.4.4]). *The optimal frame bounds $A$ and $B$ for a frame $\{\phi_i\}_{i\in\mathbb{N}}$ are given by*

$$A = \left\|S^{-1}\right\|_{\text{op}}^{-1}, \qquad B = \|S\|_{\text{op}}.$$

REMARK 3.2.7. When $S$ is compact, we can combine Lemma 3.2.3 with Theorem 3.2.6, along with standard results from functional analysis regarding the spectrum of self-adjoint, compact operators to see that $A$ and $B$ are the minimum and maximum eigenvalues of the operator $S$ in this case. In particular, this is true if $\mathcal{H}$ is any finite dimensional Hilbert space. We will use this relationship in Chapter 4 to obtain the optimal frame bounds for a family of graph signals in finite dimensional spaces. Therefore we prove the theorem only for this case.

COROLLARY 3.2.8 ([21, Theorem 1.3.1]). *Let $\{\phi_k\}_{k=1}^m$ be a frame for a finite dimensional Hilbert space $\mathcal{H}$ with frame operator $S$. Then the optimal lower and upper frame bounds are given by the smallest eigenvalue and largest eigenvalue of the matrix $S$.*

*Proof.* Let $\{\phi_k\}_{k=1}^m$ be a frame for the Hilbert space $\mathcal{H}$ of dimension $n < \infty$, with frame operator $S$. As $S$ is self-adjoint, $\mathcal{H}$ admits a basis of orthonormal eigenvectors $\{e_k\}_{k=1}^n$ of $S$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^n$. Then for $f \in \mathcal{H}$, we can write

$$Sf = \sum_{k=1}^{n} \langle f, e_k \rangle Se_k = \sum_{k=1}^{n} \lambda_k \langle f, e_k \rangle e_k.$$

Also,

$$\sum_{k=1}^{m} |\langle f, \phi_k \rangle|^2 = \langle T^* f, T^* f \rangle = \langle Sf, f \rangle = \sum_{k=1}^{n} \lambda_k |\langle f, e_k \rangle|^2.$$

Taking $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues, respectively, we have that

$$\lambda_{\min} \|f\|^2 \le \sum_{k=1}^{m} |\langle f, \phi_k \rangle|^2 \le \lambda_{\max} \|f\|^2 .$$

To see that equality is achieved, note that for $e_i$,

$$\sum_{k=1}^{m} |\langle e_i, \phi_k \rangle|^2 = \lambda_i \|e_i\|^2 .$$

□

**Stability.** In infinite dimensional Hilbert spaces, the frame operator $S$ is often not explicitly known, and can be quite difficult to compute. As such, reconstruction from the frame coefficients often requires iterative methods to approximate $S$, and the convergence rate of these iterative methods depends strongly on the frame condition number $\frac{B}{A}$. Consequently, one of the common design goals in applications of frame theory is to minimize this ratio. To illustrate what we mean by stability of a frame system, we provide a toy example showing how the lack of a lower or upper frame bound affects the representation of vectors and their reconstruction.

EXAMPLE 3.2.9. Let $I$ be some interval and $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system for $L^2(I)$. Then $\{\frac{1}{n}e_n\}_{n=1}^{\infty}$ and $\{ne_n\}_{n=1}^{\infty}$ are biorthogonal systems with each being a (Schauder) basis as

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n = \sum_{n=1}^{\infty} \underbrace{\langle f, \frac{e_n}{n} \rangle}_{c_n} ne_n = \sum_{n=1}^{\infty} \underbrace{\langle f, ne_n \rangle}_{d_n} \frac{e_n}{n}.$$

The set $\{ne_n\}_{n=1}^{\infty}$ is the original orthonormal basis with each vector stretched progressively larger. As the expansions are unique, if this formed a frame, the frame coefficients for $f \in L^2(I)$ would be given by $\{\langle f, \frac{e_n}{n} \rangle\}_{n=1}^{\infty}$. If we consider the original sequence, then each basis function has norm one in $L^2(I)$, but

$$\|Te_i\|_{\ell^2(\mathbb{N})}^2 = \sum_{n=1}^{\infty} \left| \langle e_i, \frac{e_n}{n} \rangle \right|^2 = \frac{1}{i^2}.$$

Since the limit goes to zero as $i$ tends to infinity, no lower frame bound can exist.

If we consider some error occurring at index $m$ (due to transmission of the data through a noisy channel or round-off error, for example), and we reconstruct the function $f$ as $\widetilde{f}$ from the coefficients $\{c_n\}_{n\neq m} \cup \{\widetilde{c}_m\}$, then the distance between the original and reconstructed functions in $L^2(I)$ is given by

$$\left\| f - \widetilde{f} \right\|_{L^2(I)}^2 = \| c_m m e_m - \widetilde{c}_m m e_m \|_{L^2(I)}^2 = |c_m - \widetilde{c}_m|^2 m^2.$$

So even a small difference between frame coefficients $c_m$ and $\widetilde{c}_m$ results in reconstructing a function which is not close to the original if the index $m$ is large.

Similarly, when considering the set $\left\{ \frac{e_n}{n} \right\}_{n=1}^{\infty}$, the upper frame bound fails as our sampling points cluster near the origin. Again, if we consider some difference between coefficients at index $m$, the difference between the functions in $L^2(I)$ is given by

$$\left\| f - \tilde{f} \right\|_{L^2(I)}^2 = \left\| d_m \frac{e_m}{m} - \widetilde{d}_m \frac{e_m}{m} \right\|_{L^2(I)}^2 = |d_m - \widetilde{d}_m|^2 \frac{1}{m^2}.$$

In the case of the upper frame bound failing, we can have two functions in $L^2(I)$ which are very close together in the original Hilbert space, but their coefficients in the basis grow arbitrarily far apart as the index grows. Therefore by *stability* in the frame system, we mean that for two functions which are close in the Hilbert space norm, their corresponding representations in the frame system should be relatively close, and the frame condition number gives a good indication of how much disparity is possible between these two representations.

Intuitively, then, we can think of the existence of the lower frame bound as a guarantee that the set of frame vectors contains a sufficiently dense collection of sampling points to stably reconstruct any function. Similarly the existence of the upper frame bound implies that the frame vectors are not "too crowded together" for stable reconstruction. For a formal approach to this intuition regarding density for wavelet systems (discussed in Chapter 5), we refer to the research monograph of Kutyniok [79]. For the same discussion for Gabor frames, see [22].

Example 3.2.9 provides an example of a spanning set $\left\{ \phi_i \right\}_{i\in\mathbb{N}}$ in an infinite dimensional Hilbert space that fails to be a frame, even though every $f \in \mathcal{H}$ has a unique series expansion when considering this set as a basis. This is in stark contrast to the finite dimensional setting, where every finite spanning set forms a frame.

Theorem 3.2.10 ([21, Proposition 1.1.2]). *Let* $\{\phi_n\}_{n=1}^m$ *be a sequence in a Hilbert space* $\mathcal{H}$. *Then* $\{\phi_n\}_{n=1}^m$ *is a frame for the vector space* $V := \operatorname{span}\{\phi_n\}_{n=1}^m$.

In our final example for this chapter, we demonstrate the existence of a complete set of vectors in an infinite dimensional Hilbert space for which some functions do not have *any* possible series expansions with respect to this set.

Example 3.2.11 ([21, Example 5.4.6]). Let $\mathcal{H}$ be an infinite dimensional Hilbert space with a complete orthonormal sequence $\{e_k\}_{k=1}^\infty$, and let $\{\phi_k\}_{k=1}^\infty = e_k + e_{k+1}$. We will show that this sequence is complete, Bessel (the upper frame bound holds), but that the sequence does not form a frame as there exist $f \in \mathcal{H}$ which cannot be written as a series expansion in terms of the sequence $\{\phi_k\}_{k=1}^\infty$.

*Proof.* To see that the sequence is complete, assume that $f \in \mathcal{H}$ satisfies

$$\langle f, \phi_k \rangle = 0, \qquad \forall k \in \mathbb{N}.$$

Then $\langle f, e_k \rangle + \langle f, e_{k+1} \rangle = 0$ for all $k \in N$, which implies that $|\langle f, e_k \rangle|$ is a constant. However, by Parseval's identity, we have that

$$\sum_{k=1}^\infty |\langle f, e_k \rangle|^2 = \|f\|^2,$$

which is only possible if $\langle f, e_k \rangle = 0$ for all $k$. We conclude that $f = 0$ and $\{\phi_k\}_{k=1}^\infty$ is complete.

To see that the set $\{\phi_k\}_{k=1}^{\infty}$ is a Bessel sequence, we use the standard inequality $(a+b)^2 \le 2(a^2+b^2)$, to obtain for any $f \in \mathcal{H}$

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle f, \phi_k \rangle|^2 &= \sum_{k=1}^{\infty} |\langle f, e_k \rangle + \langle f, e_{k+1} \rangle|^2 \\ &\le \sum_{k=1}^{\infty} (|\langle f, e_k \rangle| + |\langle f, e_{k+1} \rangle|)^2 \\ &\le 2\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 + 2\sum_{k=1}^{\infty} |\langle f, e_{k+1} \rangle|^2 \\ &\le 4\|f\|^2\end{aligned}$$

This provides an upper frame bound. However, the frame condition fails to hold, which can be seen by considering the functions

$$g_j := \sum_{n=1}^{j} (-1)^{n+1} e_n, \qquad \forall j \in \mathbb{N}.$$

Indeed, it is easy to see that $\|g_j\|^2 = j$ for each $j \in \mathbb{N}$, and for any fixed $j$, a straightforward calculation shows that

$$\langle g_j, \phi_k \rangle = \begin{cases} 0, & \text{if } k \neq j; \\ (-1)^{k+1} & \text{if } k = j. \end{cases}$$

Therefore

$$\sum_{k=1}^{\infty} |\langle g_j, \phi_k \rangle|^2 = |\langle g_j, \phi_j \rangle|^2 = 1 = \frac{1}{j}\|g_j\|^2.$$

As this holds for every $j \in \mathbb{N}$, we conclude the lower frame condition fails for the set $\{\phi_k\}_{k=1}^{\infty}$.

Finally, to demonstrate that there are vectors $f \in \mathcal{H}$ which do not have any series expansions with respect to the complete set $\{\phi_k\}_{k=1}^{\infty}$, consider the almost trivial example of $f = e_1$. Suppose to the contrary, that there is a sequence $\{c_k\}_{k=1}^{\infty}$ such that

$$e_1 = \sum_{k=1}^{\infty} c_k f_k = \sum_{k=1}^{\infty} c_k (e_k + e_{k+1}).$$

Then $c_1 = \langle e_1, e_1 \rangle = 1$. For $k \geq 2$, we have $0 = \langle e_1, e_k \rangle = c_{k-1} + c_k$. It follows that

$$e_1 = \sum_{k=1}^{\infty} (-1)^{k+1}(e_k + e_{k+1}).$$

However,

$$\left\| e_1 - \sum_{k=1}^{n} (-1)^{k+1}(e_k + e_{k+1}) \right\|_{\mathcal{H}}^2 = \left\| (-1)^{n+1} e_{n+1} \right\|_{\mathcal{H}}^2 = 1.$$

□

In Example 3.2.11, the operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ given by $T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k \phi_k$ is still a well-defined bounded operator into $\mathcal{H}$ by Theorem 3.2.1. However, in this case $T$ is not surjective and consequently we do not have unconditionally convergent infinite series representations in terms of the vectors $\{\phi_k\}_{k=1}^{\infty}$ for every $f \in \mathcal{H}$. This partially demonstrates why constructing frames in the case of infinite dimensional Hilbert spaces can be a highly non-trivial task.

## Chapter 4

## PART I: FRAMES FOR GRAPH SIGNALS

# 4.1 Introduction

Our contribution in this chapter is twofold. First, we propose a general framework for constructing Gabor-type frames for signals on graphs. Our approach uses general and flexible families of translation operators which include many previous constructions such as those provided in [53, 59, 94, 103]. For each such family of translations, we obtain the sharp frame bounds for the associated frames. We also provide worst-case bounds for arbitrary choices of the localizing function.

In the second part of the chapter, we examine the constructed frames in the special case where the graph $\Gamma$ is a *normal Cayley graph*. In this case, an orthonormal basis of eigenvectors of the adjacency matrix or the graph Laplacian of the graph can be explicitly obtained by exploiting the representation theory of the associated group [6]. Using this basis, we are able to study properties of our Gabor-type frames and how they relate to the structure of the underlying group: an investigation which initially began in [55].

**Previous Work on Graph Frames.** Efforts to directly generalize multiresolution and wavelet analysis to the graph setting can be found in [23, 28, 54, 60, 74, 82, 87]. Early methods to construct frames based on the eigen decomposition of the graph Laplacian are given by Coifman and Maggioni in [25] and by Maggioni and Mhaskar in [86]. In [67], Hammond, Vandergheynst and Gribonval define the graph Fourier transform and apply it to construct wavelet frames for graphs. Other examples of wavelet-type frames based on the graph Fourier transform can be found in [33, 83, 100, 104, 115]. Studies exploring fundamental limits of how efficiently signals can be represented in terms of uncertainty principles can be found in [92, 113], and a proposed fast algorithm to implement frames on graphs can be found in [75]. Some of the extensive work of defining Gabor-type frames in the graph setting, often referred to as vertex-frequency analysis, can be found in [8, 9, 102, 103, 104, 109, 111, 112]. Summaries of most of the references mentioned (and many more) can be found in the survey articles [61, 90, 105] or collected in the recent book [106].

In the present chapter we focus on the problem of constructing Gabor-type frames for signals defined on graphs. Most existing literature focuses on designing frames based on fixed choices of translation and modulation operators. In contrast, we fix the modulation operator most commonly applied in other constructions, but consider arbitrary collections of linear operators serving as translation operators. We obtain both necessary and sufficient conditions for the collection of translation and modulation acting on a fixed localization function to form a frame. Additionally, we obtain the sharp frame bounds for any construction that falls within our general framework, and compute the relevant frame bounds for several constructions which exist in the literature. To our knowledge, we are also the first to study the behavior of specific choices of translation on normal Cayley graphs and relate them back to the underlying group structure.

**Organization.** The remainder of this chapter is organized as follows. In Section 4.2 we collate all necessary definitions, notations, and background on the graph Fourier transform and graph signal frames. In Section 4.3, we include our main result: a general construction for Gabor-type frames for graph signals. In Section 4.4, we analyze the special case when the graph is Cayley. We conclude this chapter with an example which demonstrates the importance of carefully choosing the basis of eigenvectors associated to the graph in the case where it has repeated eigenvalues.

Much of this chapter represents joint work with Mahya Ghandehari and Dominique Guillot. Several of the results from Section 4.4 appear in [55], while the rest will be submitted soon [56]. Most work in Section 4.3 involved equal contributions by all authors. The work in Section 4.4 was primarily my own, but involved many helpful observations and suggestions from both advisors.

## 4.2 Notations and Definitions

Recall a graph $\Gamma$ is a pair $(V, E)$ where we assume $V = \{1, 2, 3, \ldots, N\}$ and $E \subseteq V \times V$, the Cartesian product. Then the adjacency matrix $A_\Gamma$ has entries $(i, j) = 1$ when there is an edge from vertex $i$ to vertex $j$; or equivalently, when $(i, j) \in E$ and 0 if $(i, j) \notin E$. The matrix $D_\Gamma$ is the diagonal matrix with entry $d_{i,i}$ equal to the degree of vertex $i$. A graph is *undirected* if $(i, j) \in E$ implies $(j, i) \in E$, that is, if the relation $E$ is symmetric. The following definition of the graph Fourier transform is a direct generalization of the classical Fourier transform for vectors in $\mathbb{C}^N$ introduced in Section 2.2.1.

**Graph Fourier Transform** Fix an undirected graph $\Gamma$ with $N$ vertices and choice of associated graph matrix (*i.e.*, $A_\Gamma$ or $L_\Gamma = D_\Gamma - A_\Gamma$). Let $\{\phi_j\}_{j=1}^N$ be a fixed orthonormal basis of eigenvectors for the matrix, associated to (repeated) eigenvalues $\{\lambda_j\}_{j=1}^N$. Inspired by commutative Fourier analysis, the *graph Fourier transform* was introduced by Hammond, Vandergheynst, and Gribonval in [67] as the expansion of the vector $\mathfrak{f} \in \mathbb{C}^N$ in terms of the orthonormal basis $\{\phi_j\}_{j=1}^N$. More precisely, the Fourier coefficients of $\mathfrak{f}$ are given by

$$\widehat{\mathfrak{f}}(\phi_k) = \langle\, \mathfrak{f}, \phi_k \,\rangle_{\mathbb{C}^N} = \sum_{j=1}^N \mathfrak{f}(j)\overline{\phi_k(j)}. \tag{4.1}$$

Equivalently, letting $\Phi$ be the matrix whose $j^{\text{th}}$ column is $\phi_j$, we have $\widehat{\mathfrak{f}} = \Phi^* \mathfrak{f}$. The *inverse graph Fourier transform* is then given by $\mathfrak{f} = \Phi\widehat{\mathfrak{f}}$, or

$$\mathfrak{f}(k) = \sum_{j=1}^N \widehat{\mathfrak{f}}(\phi_j)\phi_j(k). \tag{4.2}$$

See [90, 94, 95, 101] for more details on the graph Fourier transform and the associated theory.

REMARK 4.2.1. Let $\theta_N = e^{\frac{2\pi i}{N}}$ denote the first primitive $N$-th root of unity. We identify the function $\chi_k$ on $\mathbb{Z}_N$ in (2.10) with the column vector $\left(\theta_N^k, \theta_N^{2k}, \cdots, \theta_N^{(N-1)k}\right)^\top$ in $\mathbb{C}^N$, which we denote again by $\chi_k$. The set of vectors $\left\{ N^{-1/2}\chi_k \;\middle|\; k = 0, \dots, N-1 \right\}$ forms an orthonormal basis for $\mathbb{C}^N$. Applying formulas similar to (4.1) and (4.2), we obtain the graph Fourier transform from the classical formulas (2.11) and (2.12). We remark that the two transforms differ only in multiplicative factors, which arise from the different normalizations of the Haar measure.

## 4.3 Gabor Frames for Graph Signals

A natural approach to construct frames involves applying a time-frequency shift operator to a given function $g$. In Section 3.1, we introduced the work of Gabor from his seminal 1946 paper [51], where he proposes constructing such frames for functions in $L^2(\mathbb{R})$ by defining

$$g_{u,\xi}(t) := (M_\xi T_u g)(t) = g(t-u)e^{2\pi i \xi t},$$

where $(T_u g)(t) = g(t-u)$ and $(M_\xi g)(t) = e^{2\pi i \xi t} g(t)$ denote the standard translation and modulation operators on $L^2(\mathbb{R})$ from Definition 2.2.4. Such frames are commonly used in science and engineering and have been extensively studied (cf. chapters 11-14 of [21] for more details).

NOTATION 4.3.1. In this chapter, for a vector $u \in \mathbb{C}^n$, we denote an $n \times n$ diagonal matrix with diagonal entries $u_1, \dots, u_n$ as $\mathrm{diag}(u)$ or $\mathrm{diag}(u_1, \dots, u_n)$.

**Previous Gabor Frame Construction** Guided by classical commutative Fourier analysis, the work in [103] relies on extending the notions of convolution, modulation, and translation introduced in Definition 2.2.4 via the graph Fourier transform given in (4.1). To elaborate, define the *convolution* of two signals $\mathfrak{f}, \mathfrak{g}$ on a graph $\Gamma$ on $N$ vertices to be the pointwise product in the Fourier domain

$$\mathfrak{f} * \mathfrak{g} = \Phi(\widehat{\mathfrak{f}} \circ \widehat{\mathfrak{g}}), \tag{4.3}$$

where we use $\circ$ to denote entry-wise (Hadamard) multiplication of matrices. This convolution naturally leads to a notion of translation by defining

$$T_j \mathfrak{f} = \sqrt{N}(\mathfrak{f} * \delta_j) \qquad (j = 1, \ldots, N), \tag{4.4}$$

where $\delta_j$ denotes the Kronecker delta function centered at vertex $j$, *i.e.*,

$$\delta_j(k) = \delta_{j,k} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

Finally, signal modulation is defined as entrywise multiplication with the basis functions:

$$M_j \mathfrak{f} = \phi_j \circ \mathfrak{f}, \qquad \text{for } j = 1, \ldots, N. \tag{4.5}$$

Using these ideas, Shuman *et al.* [103] defined a frame for graph signals that is analogous to the classical construction of Gabor frames on the real line. Given $\mathfrak{g} : V(\Gamma) \to \mathbb{C}$, let

$$\mathfrak{g}_{j,k} := M_k T_j \mathfrak{g} \qquad \text{for } j, k = 1, \ldots, N. \tag{4.6}$$

One of the main results of [103] is the fact that, under mild assumptions, the functions $\mathfrak{g}_{j,k}$ define a frame that can be used to analyze signals on $\Gamma$.

THEOREM 4.3.2 ([103], Theorem 3). *Let* $\{\phi_j\}_{j=1}^N$ *be an orthonormal basis of eigenvectors of the graph Laplacian of* $\Gamma$ *and let* $\mathfrak{g} \in \mathbb{C}^N$. *If* $\sum_{j=1}^N \mathfrak{g}(j) \neq 0$ *then the collection of functions* $\{\mathfrak{g}_{j,k}\}_{j,k=1,\ldots,N}$ *is a frame, i.e., for all* $\mathfrak{f} \in \mathbb{R}^N$,

$$\min_{n=1}^{N}\{N \|T_n \mathfrak{g}\|_2^2\} \|\mathfrak{f}\|_2^2 \leq \sum_{j=1}^{N} \sum_{k=1}^{N} |\langle \mathfrak{f}, \mathfrak{g}_{j,k} \rangle|^2 \leq \max_{n=1}^{N}\{N \|T_n \mathfrak{g}\|_2^2\} \|\mathfrak{f}\|_2^2 \tag{4.7}$$

**General Gabor Frames for Graphs** A major difficulty that arises in the construction of Gabor frames for graph signals is the lack of a canonical notion of translation. Indeed, many notions of translations or shifts for signals on graphs have been defined in the literature, including:

1. the translation operators $T_j$ introduced by Shuman, Ricaud, and Vandergheynst [103], and defined via convolution with delta functions (see (4.4));
2. the linear isometric shift operator introduced by Girault, Gonçalves, and Fleury [59];
3. the energy-preserving shift operator introduced by Gavili and Zhang [53];
4. translation induced by the adjacency matrix of the graph, as proposed by Sandryhaila and Moura [94];
5. translation induced by pointwise multiplication with personalized PageRank vectors defined by Tepper and Sapiro [109], and;
6. the neighborhood preserving translation defined by Pasdeloup *et al.*, [62, 91].

A common feature of the above transformations is that they operate linearly on a given signal $\mathfrak{g}$. The resulting frames are therefore all instances of the following general construction, for which we obtain the sharp frame bounds.

THEOREM 4.3.3. *Let* $\{\phi_j\}_{j=1}^N$ *be an orthonormal basis of* $\mathbb{C}^N$*, let* $A_1, A_2, \ldots, A_S$ *be an arbitrary collection of complex* $N \times N$ *matrices, and let* $\mathfrak{g} \in \mathbb{C}^N$*. For* $m = 1, \ldots, S$ *and* $\ell = 1, \ldots N$*, define*

$$\mathfrak{g}_{m,\ell} := \phi_\ell \circ (A_m \mathfrak{g}), \tag{4.8}$$

*where* $\circ$ *denotes the entrywise product. Also let*

$$v = (v_k)_{k=1}^N := \sum_{j=1}^S |A_j \mathfrak{g}|^2, \tag{4.9}$$

*where the modulus and square operations are performed entrywise. Then the collection of vectors* $\left\{ \mathfrak{g}_{m,\ell} \;\middle|\; m = 1,\ldots,S, \ell = 1,\ldots,N \right\}$ *forms a frame if and only if* $v_k > 0$ *for all* $k = 1,\ldots,N$*. Moreover, in that case,*

$$A\,\|\mathfrak{f}\|_2^2 \leq \sum_{m=1}^{S}\sum_{\ell=1}^{N} |\langle\, \mathfrak{f}, \mathfrak{g}_{m,\ell}\,\rangle|^2 \leq B\,\|\mathfrak{f}\|_2^2$$

*with optimal frame bounds* $A := \min_{k=1}^{N} v_k$ *and* $B := \max_{k=1}^{N} v_k$*.*

*Proof.* For $i = 1,\ldots,N$, let $D_i$ denote the diagonal matrix with $k$-th diagonal entry equal to the $k$-th term of the vector $\phi_i$. Using that notation, observe that we have $\mathfrak{g}_{m,\ell} = D_\ell A_m \mathfrak{g}$. Now, consider the matrix whose columns are the vectors $\mathfrak{g}_{m,\ell}$:

$$T := \left( D_1A_1\mathfrak{g} \,\middle|\, D_1A_2\mathfrak{g} \,\middle|\, \ldots \,\middle|\, D_1A_S\mathfrak{g} \,\middle|\, D_2A_1\mathfrak{g} \,\middle|\, D_2A_2\mathfrak{g} \,\middle|\, \ldots \,\middle|\, D_2A_S\mathfrak{g} \,\middle|\, \ldots \,\middle|\, D_NA_S\mathfrak{g} \right).$$

By Corollary 3.2.8, the associated optimal frame bounds are given by the smallest and largest eigenvalues of $TT^*$. Using this fact, it is easy to see that the collection $\left\{ \mathfrak{g}_{m,\ell} \;\middle|\; m = 1,\ldots,S, \ell = 1,\ldots,N \right\}$ is a frame if and only if the matrix $TT^*$ is positive definite. Here, we have

$$TT^* = \sum_{i=1}^{N}\sum_{j=1}^{S} D_iA_j\mathfrak{g}\mathfrak{g}^*A_j^*D_i^*.$$

Now, observe that for any diagonal matrix $D = \mathrm{diag}(u)$ and any matrix $M$, we have $DMD^* = M \circ (uu^*)$. Hence,

$$\begin{aligned} TT^* &= \sum_{i=1}^{N}\sum_{j=1}^{S} [(A_j\mathfrak{g})(A_j\mathfrak{g})^*] \circ (\phi_i\phi_i^*) \\ &= \sum_{j=1}^{S} [(A_j\mathfrak{g})(A_j\mathfrak{g})^*] \circ \left(\sum_{i=1}^{N} \phi_i\phi_i^*\right) \\ &= \sum_{j=1}^{S} [(A_j\mathfrak{g})(A_j\mathfrak{g})^*] \circ I_N, \end{aligned}$$

where the last line follows from the fact that the $\phi_i$'s form an orthonormal basis of $\mathbb{C}^N$. Hence $TT^*$ is diagonal with diagonal entries given by $\sum_{j=1}^{S} |A_j\mathfrak{g}|^2$. The result now follows immediately from Corollary 3.2.8. □

COROLLARY 4.3.4. *In the same setting as Theorem 4.3.3, the family*

$$\left\{ \mathfrak{g}_{m,\ell} \;\middle|\; m = 1, \ldots, S, \ell = 1, \ldots, N \right\}$$

*forms a frame if and only if for every* $1 \le k \le N$ *there exists* $1 \le j \le S$ *such that* $(A_j \mathfrak{g})_k \neq 0$.

While Theorem 4.3.3 provides explicit frame bounds for (4.8), it is not immediately clear how the entries of the vector $v$ in (4.9) vary with the vector $\mathfrak{g}$. The following result provides a different description of the entries of $v$ that clarifies this relationship.

THEOREM 4.3.5. *Consider the same setting as Theorem 4.3.3 with* $A_j := \big(a_{k,\ell}^{(j)}\big)_{k,\ell=1}^N$ *and* $v \in \mathbb{C}^N$ *as in (4.9). For* $k, \ell = 1, \ldots, N$, *define* $\mathbf{w}_{k,\ell} \in \mathbb{C}^S$ *by*

$$\mathbf{w}_{k,\ell} := \big(a_{k,\ell}^{(j)}\big)_{j=1}^S,$$

*and let*

$$C_k := \big(\langle\, \mathbf{w}_{k,\ell}, \mathbf{w}_{k,m} \,\rangle\big)_{\ell,m=1}^N = \left( \sum_{j=1}^S a_{k\ell}^{(j)} \overline{a_{km}^{(j)}} \right)_{\ell,m=1}^N \in \mathbb{C}^{N \times N}. \tag{4.10}$$

*Then* $v_k = \mathfrak{g}^* C_k \mathfrak{g}$ *for any* $1 \le k \le N$. *In particular, the family of functions* $\{\mathfrak{g}_{m,\ell}\}$ *forms a frame if and only if* $\mathfrak{g} \notin \bigcup_{k=1}^N \ker C_k$.

*Proof.* The $k^{\text{th}}$ entry of $v$ is given by

$$\begin{aligned}
v_k &= \sum_{j=1}^S \left| \sum_{\ell=1}^N a_{k\ell}^{(j)} \mathfrak{g}_\ell \right|^2 = \sum_{j=1}^S \left( \sum_{\ell=1}^N a_{k\ell}^{(j)} \mathfrak{g}_\ell \right) \overline{\left( \sum_{m=1}^N a_{km}^{(j)} \mathfrak{g}_m \right)} \\
&= \sum_{j=1}^S \sum_{\ell,m=1}^N a_{k\ell}^{(j)} \overline{a_{km}^{(j)}} \mathfrak{g}_\ell \overline{\mathfrak{g}_m} \\
&= \sum_{\ell,m=1}^N \overline{\mathfrak{g}_m} \,\langle\, \mathbf{w}_{k\ell}, \mathbf{w}_{km} \,\rangle\, \mathfrak{g}_\ell \\
&= \mathfrak{g}^* C_k \mathfrak{g},
\end{aligned}$$

where $C_k$ is as in (4.10). ∎

Note that, for each $k$, the matrix $C_k$ in Theorem 4.3.5 is the Gram matrix generated by the vectors $\{\mathbf{w}_{k,j}\}_{j=1}^N$, therefore each is a positive semidefinite Hermitian matrix. Using Theorem 4.3.5, we immediately obtain useful estimates on the frame bounds given in Theorem 4.3.3, as well as the resulting condition number of the frame. Given a Hermitian matrix $M$, denote by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ the smallest and largest eigenvalues of $M$, respectively. The following result provides estimates on the frame bounds that are independent of $\mathfrak{g}$.

Corollary 4.3.6. *Consider the same setting as Theorem 4.3.3, and assume furthermore that* $\|\mathfrak{g}\| = 1$. *Then*

$$\min_{k=1}^{N} \lambda_{\min}(C_k) \cdot \|\mathfrak{f}\|_2^2 \le \sum_{m=1}^{S} \sum_{\ell=1}^{N} |\langle \mathfrak{f}, \mathfrak{g}_{m,\ell} \rangle|^2 \le \max_{k=1}^{N} \lambda_{\max}(C_k) \cdot \|\mathfrak{f}\|_2^2,$$

*In particular, if* $\min_{k=1}^N \lambda_{\min}(C_k) > 0$, *then* $\mathcal{G} := \{ \mathfrak{g}_{m,\ell} \mid m = 1, \ldots, S, \ell = 1, \ldots, N \}$ *forms a frame whose condition number* $\kappa(\mathcal{G})$ *satisfies*

$$\kappa(\mathcal{G}) \le \frac{\max_{k=1}^N \lambda_{\max}(C_k)}{\min_{k=1}^N \lambda_{\min}(C_k)}.$$

We can further refine the analysis of the condition number for the frame $\mathcal{G}$ given in Corollary 4.3.6 as follows. Recall that $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of the matrix pencil $A - zB$ if

$$\det(A - \lambda B) = 0.$$

In that case, there exists $v \in \mathbb{C}^N \backslash \{0\}$ such that $Av = \lambda Bv$. When $B$ is positive definite, the case we will focus on below, the eigenvalue problem for the pencil is equivalent to the standard Hermitian eigenvalue problem

$$B^{-1/2} A B^{-1/2} x = \lambda x. \tag{4.11}$$

If $v$ is an eigenvector of the pencil $A - \lambda B$, then $x = B^{1/2} v$ is an eigenvector of the modified problem (4.11) corresponding to the same eigenvalue. As a consequence, the pencil has exactly $n$ real eigenvalues $\lambda_1 \le \cdots \le \lambda_N$ that can be computed via the Courant-Fischer min-max principles:

$$\lambda_j = \min_{\dim U = j} \max_{u \in U} \frac{u^* A u}{u^* B u} = \max_{\dim U = N-j+1} \min_{u \in U} \frac{u^* A u}{u^* B u}.$$

In particular,

$$\lambda_1 = \min_{u \in \mathbb{C}^N} \frac{u^* A u}{u^* B u} \qquad \lambda_N = \max_{u \in \mathbb{C}^N} \frac{u^* A u}{u^* B u}. \tag{4.12}$$

See, *e.g.*, [73, 84] for more details about matrix pencils.

As a consequence of the above, we immediately obtain the following sharp upper bound on the condition number of the frame $\{\mathfrak{g}_{m,\ell}\}$ under the assumption that $\|\mathfrak{g}\| = 1$. For two Hermitian positive definite matrices $A, B$, let $\lambda_{\min}(A, B)$ and $\lambda_{\max}(A, B)$ denote the smallest and largest eigenvalues of the pencil $A - zB$ respectively.

THEOREM 4.3.7. *Let* $\mathcal{G} := \{ \mathfrak{g}_{m,\ell} \mid m = 1, \ldots, S, \ell = 1, \ldots, N \}$ *with* $\mathfrak{g}_{m,\ell}$ *as in Theorem 4.3.3. Let* $C_k$ *be as in* (4.10) *and assume* $C_1, \ldots, C_N$ *are positive definite. Then*

$$\sup_{\|\mathfrak{g}\|=1} \kappa(\mathcal{G}) = \max_{k,\ell=1,\ldots,N} \lambda_{\max}(C_k, C_\ell). \tag{4.13}$$

*Equality is attained when* $\mathfrak{g}$ *is an eigenvector associated to the generalized eigenvalue problem* $C_{k^*} - \lambda C_{\ell^*}$*, where* $k^*$ *and* $\ell^*$ *are values of* $k$ *and* $\ell$ *attaining the maximum in* (4.13).

*Proof.* By Theorems 4.3.3 and 4.3.5, we have

$$\kappa(\mathcal{G}) = \frac{\max_{k=1}^N \mathfrak{g}^* C_k \mathfrak{g}}{\min_{\ell=1}^N \mathfrak{g}^* C_\ell \mathfrak{g}} = \max_{k,\ell=1,\ldots,N} \frac{\mathfrak{g}^* C_k \mathfrak{g}}{\mathfrak{g}^* C_\ell \mathfrak{g}}.$$

The result follows from (4.12) upon maximizing over $\mathfrak{g}$. □

### 4.3.1 Applications to Special Translation Operators

For signals defined on graphs, several translation operators that have been proposed in the literature share the common feature that they operate by entry-wise multiplication in the Fourier domain, *i.e.*,

$$\widehat{T\mathfrak{g}} = \widehat{\mathfrak{f}} \circ \widehat{\mathfrak{g}}$$

for some $\mathfrak{f} \in \mathbb{C}^N$. That is, they can be realized as Fourier multipliers. A *Fourier multiplier* of a locally compact abelian group is a function $T$ associated with the map $m : \widehat{G} \to \mathbb{C}$ and can be defined via the formula

$$\widehat{Tf}(\xi) := m(\xi)\widehat{f}(\xi)$$

for all $f \in L^1(G)$. Using the notion of convolution defined in (4.3), this is equivalent to $T\mathfrak{g} = \mathfrak{f} * \mathfrak{g}$. Equivalently, for a given vector $w \in \mathbb{C}^N$, denote by $D_w$ the diagonal matrix with diagonal entries $w_1, w_2, \ldots, w_N$. Then the above operator can be written

$$T = \Phi D_{\widehat{f}} \Phi^*. \tag{4.14}$$

The observation that graph translation as defined in (4.4) could be written as a Fourier multiplier was originally published in [7], but we also consider several other examples below. In fact, the first four examples of translation or shift given at the beginning of this section are special instances of this construction. We remark that some proposed translations, such as pointwise multiplication with vectors in the vertex domain [109], can not be written as Fourier multipliers. However, when using polynomials of the adjacency matrix as in [94], this is obviously a Fourier multiplier as for any polynomial $P \in \mathbb{C}[x]$,

$$T_P := P(A) = P(\Phi D_\lambda \Phi^*) = \Phi P(D_\lambda)\Phi^*,$$

where $D_\lambda$ is the diagonal matrix with entries $\{\lambda_i\}_{i=1}^N$. We briefly summarize how the first three examples of translations manifest as Fourier multipliers below.

**Example 1.** In [101, 102, 103], the translation operator $T_i$ is defined by convolution with $\sqrt{N}\delta_i$. As in (4.14), this convolution can be written as

$$T_i = \Phi D_{\sqrt{N}\widehat{\delta_i}} \Phi^*.$$

**Example 2.** The isometric shift operator given by [59] can be realized as the Fourier multiplier

$$\widehat{Tf}(\phi_k) = \exp\left(\frac{-i\pi\sqrt{\lambda_k}}{\sqrt{\rho}}\right)\widehat{f}(\phi_k),$$

where $\rho$ is chosen to be upper bound on the largest eigenvalue of the graph Laplacian. Consequently, when using the basis of eigenvectors for the Laplacian, $\sqrt{\frac{\lambda_k}{\rho}} \in [0,1]$.

**Example 3.** Taking a similar approach to [59], [53] defines two translation operators

$$\widehat{T_1 f}(\phi_k) = \exp(i\omega_k)\, \widehat{f}(\phi_k), \qquad \widehat{T_2 f}(\phi_k) = \exp\left(i\left(c - \frac{2\pi(k-1)}{N}\right)\right) \widehat{f}(\phi_k),$$

where $\omega_k \in [0, 2\pi)$ is chosen arbitrarily such that $\omega_k \neq \omega_r$ for $k \neq r$ and $c$ is a constant phase shift term. The operator $T_2$ is a special case of $T_1$, chosen so that (when $c = 0$) $T_2^N = I_N$, and some standard orthogonality relations from discrete Fourier analysis can be applied. This is also the translation used in [111].

The following result provides explicit frame bounds when translations can be realized as Fourier multipliers.

THEOREM 4.3.8. *Let $\{\phi_j\}_{j=1}^N$ be an orthonormal basis of $\mathbb{C}^N$ and define the vectors $\mu_j := \left(\phi_k(j)\right)_{k=1}^N \in \mathbb{C}^N$, that is*

$$\left( \phi_1 \,\middle|\, \ldots \,\middle|\, \phi_N \right) = \begin{pmatrix} \mu_1 \\ \hline \vdots \\ \hline \mu_N \end{pmatrix}.$$

*Let $\{f_j\}_{j=1}^S$ be an arbitrary collection of vectors in $\mathbb{C}^N$, and let $\mathfrak{g} \in \mathbb{C}^N$. Define*

$$A_i = \Phi D_{f_i} \Phi^*, \qquad \text{for } i = 1, \ldots, S.$$

*For $1 \leq m \leq S, 1 \leq \ell \leq N$, define $\mathfrak{g}_{m,\ell}$ as in (4.8). Let $F$ be the $N \times S$ matrix whose $j$-th columns is $f_j$. Then we have*

$$A\,\|\mathfrak{f}\|_2^2 \leq \sum_{m=1}^{S} \sum_{\ell=1}^{N} |\langle \mathfrak{f}, \mathfrak{g}_{m,\ell} \rangle|^2 \leq B\,\|\mathfrak{f}\|_2^2,$$

*where*

$$A = \min_{k=1}^{N} \|F^*(\mu_k \circ \overline{\widehat{\mathfrak{g}}})\|^2 \qquad \text{and} \qquad B = \max_{k=1}^{N} \|F^*(\mu_k \circ \overline{\widehat{\mathfrak{g}}})\|^2.$$

*The constants $A$ and $B$ are sharp.*

*Proof.* We compute the vector $v$ in Theorem 4.3.3. We have

$$v = \sum_{i=1}^{N} |A_i \mathfrak{g}|^2 = \sum_{i=1}^{N} |\Phi D_{f_i} \Phi^* \mathfrak{g}|^2 = \sum_{i=1}^{N} |\Phi(f_i \circ \widehat{\mathfrak{g}})|^2.$$

Let $F = \left( f_1 \middle| \dots \middle| f_S \right)$ be the matrix whose columns are $f_1, \dots, f_S$. Then

$$\begin{aligned} v_k = \sum_{i=1}^{S} |\langle \mu_k, f_i \circ \widehat{\mathfrak{g}} \rangle|^2 &= \sum_{i=1}^{S} \left| \langle f_i, \mu_k \circ \overline{\widehat{\mathfrak{g}}} \rangle \right|^2 \\ &= \sum_{i=1}^{S} \langle f_i, \mu_k \circ \overline{\widehat{\mathfrak{g}}} \rangle \overline{\langle f_i, \mu_k \circ \overline{\widehat{\mathfrak{g}}} \rangle} \\ &= (\mu_k \circ \overline{\widehat{\mathfrak{g}}})^* F F^* (\mu_k \circ \overline{\widehat{\mathfrak{g}}}) \\ &= \| F^* (\mu_k \circ \overline{\widehat{\mathfrak{g}}}) \|^2. \end{aligned}$$

The result now follows from Theorem 4.3.3. □

COROLLARY 4.3.9. *Assume the functions* $\{ f_j \}_{j=1}^{N}$ *are orthonormal in Theorem 4.3.8. Then the family of functions* $\{ \mathfrak{g}_{m,\ell} \mid m, \ell = 1, \dots, N \}$ *forms a frame if and only if for every* $1 \le k \le N$ *there exists* $1 \le j \le N$ *such that* $\phi_j(k)$ *and* $\widehat{\mathfrak{g}}(j)$ *are both non-zero. In that case, the sharp bounds for the associated frame are given by*

$$A = \min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2 \qquad \textit{and} \qquad B = \max_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2.$$

In particular, observe that the frame is tight when $\widehat{\mathfrak{g}}$ is constant.

COROLLARY 4.3.10. *Assume the functions* $\{ f_j \}_{j=1}^{N}$ *are orthonormal in Theorem 4.3.8. Moreover, assume* $\widehat{\mathfrak{g}}$ *is constant. Then the functions* $\{ \mathfrak{g}_{m,\ell} \mid m, \ell = 1, \dots, N \}$ *form a tight frame.*

*Proof.* Assume $\widehat{\mathfrak{g}} \equiv c$ for some $c \in \mathbb{C}$. Then

$$A = |c|^2 \min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 = |c|^2$$

since the $\{\phi_j\}_{j=1}^{N}$ are orthonormal. Similarly, we obtain $B = |c|^2$. □

Interestingly, the frame bounds in Corollary 4.3.9 are independent of the choice of the vectors $\{f_i\}_{i=1}^N$, as long as they are orthonormal. Using a trivial estimate on the above optimal frame bound and the orthonormality of the rows of $\Phi$, we immediately obtain the following estimates that are independent of the basis $\{\phi_j\}_{j=1}^N$.

COROLLARY 4.3.11. *Under the assumptions of Theorem 4.3.8, we have*

$$A = \min_{k=1}^{N} \|F^*(\mu_k \circ \overline{\widehat{\mathfrak{g}}})\|^2 \geq \lambda_{\min}(FF^*) \cdot \min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2 \geq \lambda_{\min}(FF^*) \cdot \min_{k=1}^{N} |\widehat{\mathfrak{g}}(k)|^2,$$

$$B = \max_{k=1}^{N} \|F^*(\mu_k \circ \overline{\widehat{\mathfrak{g}}})\|^2 \leq \lambda_{\max}(FF^*) \cdot \max_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2 \leq \lambda_{\max}(FF^*) \cdot \max_{k=1}^{N} |\widehat{\mathfrak{g}}(k)|^2.$$

*In particular, if $FF^*$ is non-singular and $\min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2 > 0$, then the collection of vectors $\mathcal{G} := \left\{ \mathfrak{g}_{m,\ell} \;\middle|\; 1 \leq m \leq S, 1 \leq \ell \leq N \right\}$ forms a frame whose condition number satisfies:*

$$\kappa(\mathcal{G}) \;\leq\; \kappa(FF^*)\, \frac{\max_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2}{\min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2} \;\leq\; \kappa(FF^*)\, \frac{\max_{k=1}^{N} |\widehat{\mathfrak{g}}(k)|^2}{\min_{k=1}^{N} |\widehat{\mathfrak{g}}(k)|^2},$$

*where $\kappa(FF^*) := \|FF^*\|\,\|(FF^*)^{-1}\|$ denotes the condition number of the matrix $FF^*$.*

We illustrate how Theorem 4.3.8 can be applied to yield the sharp frame bounds for the frame constructed using popular translation operators.

COROLLARY 4.3.12. *Let $\{\,\phi_j\,\}_{j=1}^N$ be an arbitrary orthonormal basis of $\mathbb{C}^N$ and let $\mathfrak{g} \in \mathbb{C}^N$. For $i = 1, \ldots, N$, define*

$$T_i\mathfrak{g} := \mathfrak{g} * (\sqrt{N}\delta_i). \tag{4.15}$$

*For $m, \ell = 1, \ldots, N$, define $\mathfrak{g}_{m,\ell}$ as in (4.8). Then we have*

$$A\,\|\mathfrak{f}\|_2^2 \;\leq\; \sum_{m=1}^{N} \sum_{\ell=1}^{N} |\langle\, \mathfrak{f}, \mathfrak{g}_{m,\ell}\,\rangle|^2 \;\leq\; B\,\|\mathfrak{f}\|_2^2$$

*where*

$$A = N \cdot \min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2 \qquad \text{and} \qquad B = N \cdot \max_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2 \cdot |\widehat{\mathfrak{g}}(j)|^2.$$

*The constants $A$ and $B$ are sharp.*

*Proof.* Recall that $T_i\mathfrak{g} = \Phi D_{\sqrt{N}\widehat{\delta_i}} \Phi^* \mathfrak{g}$. Now, for $k = 1, \dots, N$, we have

$$\widehat{\delta_i}(k) = \langle\, \delta_i, \phi_k \,\rangle = \sum_{j=1}^{N} \delta_i(j)\overline{\phi_k(j)} = \overline{\phi_k(i)}.$$

It follows easily that $\widehat{\delta_1}, \dots, \widehat{\delta_N}$ is an orthonormal basis of $\mathbb{C}^N$. The result now follows immediately from Corollary 4.3.9. □

Finally, we apply Theorem 4.3.8 to obtain the sharp bounds for translation by repeatedly applying the energy preserving shift operator of Gavili and Zhang [53]. This is the operator $T_2$ defined in the graph spectral domain in Example 3 above. Let $A = \Phi\Lambda\Phi^*$ denote the eigen decomposition of the adjacency matrix $A$ of the graph $\Gamma$. The authors in [53] define the shift operator $A_\alpha$ by

$$A_\alpha = \Phi D_\alpha \Phi^*, \tag{4.16}$$

where $\alpha \in \mathbb{C}^N$ is an arbitrary vector of distinct complex numbers of modulus 1. Of particular interest is the case where $\alpha_k\overline{\alpha_\ell} = e^{-i\frac{2\pi(k-\ell)}{N}}$, *i.e.*,

$$\alpha_k = e^{i\left(c - \frac{2\pi(k-1)}{N}\right)} \tag{4.17}$$

where $c \in [0, 2\pi)$. Note that [53] only considers the case when $c = 0$ and observe that, under this assumption, the shift operator $A_\alpha$ given by (4.16) satisfies $A_\alpha^N = I$.

Corollary 4.3.13. *Let $\{\phi_j\}_{j=1}^{N}$ be an arbitrary, orthonormal basis of $\mathbb{C}^N$ and let $\mathfrak{g} \in \mathbb{C}^N$. For $i = 1, \dots, N$, define*

$$A_i\mathfrak{g} := A_\alpha^{i-1}\mathfrak{g} \qquad i = 1, \dots, N, \tag{4.18}$$

*with $\alpha \in \mathbb{C}^N$ as in (4.17). For $m, \ell = 1, \ldots, N$, define $\mathfrak{g}_{m,\ell}$ as in (4.8). Then we have*

$$A\,\|\mathfrak{f}\|_2^2 \le \sum_{m=1}^{N}\sum_{\ell=1}^{N} |\langle\, \mathfrak{f}, \mathfrak{g}_{m,\ell}\,\rangle|^2 \le B\,\|\mathfrak{f}\|_2^2$$

*where*

$$A = N \cdot \min_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2\, |\widehat{\mathfrak{g}}(j)|^2 \qquad \textit{and} \qquad B = N \cdot \max_{k=1}^{N} \sum_{j=1}^{N} |\phi_j(k)|^2\, |\widehat{\mathfrak{g}}(j)|^2.$$

*The constants $A$ and $B$ are sharp.*

*Proof.* Let $\alpha = \big(\alpha_k\big)_{k=1}^{N} \in \mathbb{C}^N$ be given by

$$\alpha_k = e^{i\left(c - \frac{2\pi(k-1)}{N}\right)}$$

for some $c \in [0, 2\pi)$. For $1 \le j \le N$, let

$$f_i := (\alpha_1^{i-1}, \alpha_2^{i-1}, \ldots, \alpha_N^{i-1})^\top.$$

Observe that $A_i = \Phi D_{f_i} \Phi^*$. Now, for $1 \le k, \ell \le N$, we have

$$\begin{aligned}\langle\, f_k, f_\ell\,\rangle &= \sum_{j=1}^{N} \alpha_j^{k-1}\overline{\alpha_j^{\ell-1}} = \sum_{j=1}^{N} e^{-(k-1)i\frac{2\pi(j-1)}{N} + (\ell-1)i\frac{2\pi(j-1)}{N}} \\ &= \sum_{j=1}^{N} e^{-i(\ell-k)\frac{2\pi(j-1)}{N}} = \sum_{j=1}^{N} \theta_N^{(k-\ell)(j-1)}\end{aligned}$$

where $\theta_N$ is the root of unity defined in Remark 4.2.1. Using the standard orthogonality relations for the discrete Fourier transform, we conclude that

$$\langle\, f_k, f_\ell\,\rangle = \begin{cases} N & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from Corollary 4.3.9 after rescaling the functions $\big\{f_i\big\}_{i=1}^{N}$. $\square$

## 4.4 Discrete Frames for Cayley Graphs

We now examine how Gabor-type frames behave for signals defined on Cayley graphs. Given a finite (not necessarily abelian) group $G$ and a subset $S \subset G$, the

*Cayley graph* $\mathrm{Cay}(G; S)$ is the graph whose vertex set is given by the elements of $G$, with adjacency defined as $(x, y) \in E$ if and only if $x^{-1}y \in S$. If $S$ is symmetric (*i.e.*, $S^{-1} = S$) then the graph is undirected. A Cayley graph is called *normal* if $S$ is closed under conjugation (*i.e.*, $gSg^{-1} = S$ for all $g \in G$). Observe that Cayley graphs are regular of degree $|S|$. As a result, the eigenvectors of both the adjacency and Laplacian matrices of a Cayley graph are the same, so the following analysis applies to either choice of analyzing matrix. For the remainder of this chapter, we assume that $\Gamma = \mathrm{Cay}(G; S)$ is the Cayley graph of a finite group $G$ of order $N$.

The adjacency matrix for a Cayley graph can be indexed by the group elements. For this purpose, we assume that any group has been given an arbitrary, but fixed, ordering. In this case, the vector $e_g$ refers to the standard basis vector which is 1 in the $g$-th position, and 0 elsewhere. Under this assumption, we have the following lemma.

LEMMA 4.4.1 (cf. [6, Corollary 3.2]). *The adjacency matrix for* $\Gamma = \mathrm{Cay}(G; S)$ *is given by*

$$A_\Gamma = \sum_{s \in S} R(s)$$

*where $R$ is the right regular representation of the group as described in Example 2.1.2.*

*Proof.* Noting that

$$\begin{aligned} \langle\, R(s)e_y, e_x \,\rangle &= \sum_{g \in G} \overline{e_x(g)} R(s) e_y(g) = \sum_{g \in G} \overline{e_x(g)} e_y(gs) \\ &= \sum_{g \in G} \overline{e_x(g)} e_{ys^{-1}}(g) = \begin{cases} 1, & \text{if } x^{-1}y = s, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then $R(s)$ is the adjacency matrix of $\mathrm{Cay}(G; \{s\})$. The general case follows easily with the additional observation that $\sum_{g \in G} R(g)$ is the matrix of all ones. □

Lemma 4.4.1 shows a clear link between the representation theory discussed in Chapter 2 and Cayley graphs. In fact, one major advantage of working with normal Cayley graphs is that their (adjacency or Laplacian) eigenvectors can be written explicitly via the representation theory of the associated group. As $G$ is finite, we let $\widehat{G} := \{\pi^{(i)}\}_{i=1}^{D}$ be the collection of all irreducible representations of $G$. Recall that $\pi_{i,j}^{(k)}(g) = \langle \pi^{(k)} e_j, e_i \rangle$, and we will slightly abuse notation to also denote $\pi_{i,j}^{(k)} := \sum_{g\in G} \pi_{i,j}^{(k)}(g) e_g$ as a vector, where usage should be clear from context. Let $\chi_\pi(g)$ denote the trace of matrix $\pi(g)$, which is the *character* of the representation $\pi$.

Then for each $1 \le k \le D$ and $1 \le i, j \le d_\pi$, the vector $\pi_{i,j}^{(k)}$ is an eigenvector for all normal Cayley graphs of the group $G$. The associated eigenvalues can be written explicitly in closed form, for which we will use the following lemma about class functions. We say $f : G \to \mathbb{C}$ is a *class function* if $f(xgx^{-1}) = f(g)$ for all $x, g \in G$.

LEMMA 4.4.2 ([99, Chapter 2, Proposition 6]). *Let $G$ be a finite group. Let $f : G \to \mathbb{C}$ be a class function and $\rho : G \to \mathcal{U}(\mathcal{H}_\rho)$ be an irreducible representation of $G$. Define the linear map*

$$\widetilde{\rho}(f) = \sum_{g\in G} f(g)\rho(g).$$

*Then we have the formula*

$$\widetilde{\rho}(f) = \frac{1}{d_\rho}\langle f, \overline{\chi}_\rho \rangle I,$$

*where $I$ denotes the identity on $\mathcal{H}_\rho$.*

*Proof.* First note that for all $g \in G$ we have

$$\rho(g)\widetilde{\rho}(f)\rho(g)^{-1} = \rho(g)\left[\sum_{h\in G} \rho(h)\right]\rho(g)^{-1} = \sum_{h\in G} f(h)\rho(ghg^{-1}) = \sum_{h\in G} f(g^{-1}hg)\rho(h) = \widetilde{\rho}(f),$$

thus $\widetilde{\rho} \in \mathcal{I}(\rho)$. Then we can write

$$\widetilde{\rho}(f) = \frac{1}{|G|}\sum_{g\in G} \rho(g)\widetilde{\rho}(f)\rho(g)^{-1} = c_\rho I$$

for some constant $c_\rho$, where the first equality is from taking $|G|$ copies of $\widetilde{\rho}$, and the second follows by Schur's lemma (Theorem 2.1.1). By taking the trace of each side, we see that $\mathrm{Tr}(c_\rho I) = d_\rho c_\rho$, and

$$\mathrm{Tr}(\widetilde{\rho}(f)) = \sum_{g\in G} f(g)\,\mathrm{Tr}(\rho(g)) = \sum_{g\in G} f(g)\chi_\rho(g) = \langle f, \overline{\chi_\rho} \rangle.$$

Solving for $c_\rho$ yields the result. □

Theorem 4.4.3 ([15, Proposition 6.3.1]). *Let $\Gamma = \mathrm{Cay}(G;S)$ be the Cayley graph of a finite group $G$ and assume $S$ is closed under conjugation. Then for each $\pi \in \widehat{G}$ and any $1 \le i,j \le d_\pi$, we have*

$$A_\Gamma\, \pi_{i,j} = \frac{1}{d_\pi}\left(\sum_{s\in S}\chi_\pi(s)\right)\pi_{i,j}.$$

*Proof.* Let $\mathbb{1}_S$ denote the indicator function of $S$. By Lemma 4.4.1, we have that

$$A_\Gamma = \sum_{s\in S} R(s) = \sum_{g\in G}\mathbb{1}_S(g)R(g).$$

Recall that by the Peter-Weyl theorem (Theorem 2.1.6), the right regular representation $R$ is unitarily equivalent to $\bigoplus_{\pi\in\widehat{G}} d_\pi\pi$, *i.e.*, there exists a unitary matrix $U$ so that

$$R(g) = U^*\Big(\bigoplus_{\pi\in\widehat{G}} d_\pi\pi(g)\Big)U,\ \forall g\in G.$$

Define the subspace $\mathcal{E}_{\pi,i} := \mathrm{span}\big\{\ \pi_{i,j}\ \big|\ 1\le j\le d_\pi\ \big\}$. Applying Theorem 2.1.6 again, we have $\mathcal{E}_{\pi,i}$ is an invariant subspace of $R$ such that, for $U$ restricted to this subspace, $U^*\pi(g)U = R^{\mathcal{E}_{\pi,i}}(g)$ for all $g\in G$, where $R^{\mathcal{E}_{\pi,i}}$ denotes the restriction of the operator $R$ to $\mathcal{E}_{\pi,i}$. Hence,

$$\begin{aligned}
A\pi_{i,j} &= \sum_{g\in G}\mathbb{1}_S(g)R^{\mathcal{E}_{\pi,i}}(g)\pi_{i,j} = \sum_{g\in G}\mathbb{1}_S(g)U^*\pi(g)U\pi_{i,j}\\
&= \sum_{g\in G}\mathbb{1}_S(g)U^*\pi(g)e_j = \sum_{k=1}^{d_\pi}\sum_{g\in G}\mathbb{1}_S(g)\pi_{k,j}(g)\pi_{i,k}\\
&= \sum_{k=1}^{d_\pi}\frac{1}{d_\pi}\langle\,\mathbb{1}_S,\overline{\chi_\pi}\,\rangle\delta_{k,j}\pi_{i,k} = \left[\frac{1}{d_\pi}\sum_{g\in S}\chi_\pi(g)\right]\pi_{i,j},
\end{aligned}$$

Where the final line follows from Lemma 4.4.2 and the fact that $\mathbb{1}_S$ is a class function when $S$ is the union of conjugacy classes. ☐

For a finite group $G$, the Schur orthogonality relations (Theorem 2.1.5) can be stated as:

$$\sum_{g\in G} \overline{\pi^{(j)}_{n,m}(g)}\, \pi^{(k)}_{n',m'}(g) = \delta_{j,k}\delta_{n,n'}\delta_{m,m'} \frac{N}{d_{\pi^{(j)}}}. \tag{4.19}$$

As a consequence, the following scaled coefficient functions

$$\phi^{(k)}_{i,j} := \sqrt{\frac{d_\pi}{N}}\, \pi^{(k)}_{i,j} \tag{4.20}$$

are an orthonormal basis of eigenvectors for normal Cayley graphs. In particular, note that we do not assume the graph is undirected in this section, as normal Cayley graphs are not necessarily generated from inverse closed sets. However, Theorem 4.4.3 shows that these graphs are always diagonalized by the basis of coefficient functions of irreducible unitary representations of the underlying group.

### 4.4.1 Frame Bounds for Normal Cayley Graphs

We now revisit the frame construction given in Theorem 4.3.3, in the case where $\Gamma$ is a normal Cayley graph. Recall the important special case where the translation operators are diagonal in the orthonormal basis $\{\phi_j\}_{j=1}^N$. In particular, Corollary 4.3.10 shows that a tight frame is always obtained when the multipliers $\{f_j\}_{j=1}^N$ are orthonormal and $\widehat{\mathfrak{g}}$ is constant, *i.e.*, $\mathfrak{g} = c\sum_{n=1}^N \phi_n$ for some $c \in \mathbb{C}$. We now show how the last assumption can be considerably relaxed when working on normal Cayley graphs, by exploiting the supplementary structure of the group representations, to obtain a broader class of functions $\widehat{\mathfrak{g}}$ which guarantee tight frames.

Theorem 4.4.4. *Let $\Gamma = \mathrm{Cay}(G; S)$ be the Cayley graph of a finite group $G$ of order $N$ with $S$ closed under conjugation. Assume $\widehat{\mathfrak{g}}$ is constant over every representation of $G$, i.e., for every $\pi \in \widehat{G}$,*

$$\widehat{\mathfrak{g}}(\pi_{i,j}) = \widehat{\mathfrak{g}}_\pi$$

*for some constant* $\widehat{\mathfrak{g}}_\pi \in \mathbb{C}$ *that is independent of* $i$ *and* $j$*. Let* $\{f_j\}_{j=1}^N$ *be an orthonormal basis of* $\mathbb{C}^N$ *and define*

$$A_i = \Phi D_{f_i} \Phi^*, \qquad (i = 1, \dots, N).$$

*Then the family of functions* $\{\, \mathfrak{g}_{m,\ell} \;\big|\; m, \ell = 1, \dots, N \,\}$ *defined in* (4.8) *forms a tight frame with optimal frame bounds* $A = B = \frac{1}{N} \sum_{\pi \in \widehat{G}} |\widehat{\mathfrak{g}}_\pi|^2 d_\pi^2$.

*Proof.* We compute the frame bounds given in Corollary 4.3.9. We have

$$\begin{aligned}
\sum_{t=1}^{N} |\phi_t(k)|^2 \cdot |\widehat{\mathfrak{g}}(t)|^2 &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} \frac{d_\pi}{N} |\pi_{i,j}(k)|^2 |\widehat{\mathfrak{g}}(\pi_{i,j})|^2 \\
&= \frac{1}{N} \sum_{\pi \in \widehat{G}} d_\pi |\widehat{\mathfrak{g}}_\pi|^2 \sum_{i,j=1}^{d_\pi} |\pi_{i,j}(k)|^2 \\
&= \frac{1}{N} \sum_{\pi \in \widehat{G}} d_\pi |\widehat{\mathfrak{g}}_\pi|^2 \sum_{i,j=1}^{d_\pi} \pi_{i,j}(k) \pi_{j,i}(k^{-1}) \\
&= \frac{1}{N} \sum_{\pi \in \widehat{G}} d_\pi |\widehat{\mathfrak{g}}_\pi|^2 \sum_{n=1}^{d_\pi} [\pi(k)\pi(k^{-1})]_{i,i} \\
&= \frac{1}{N} \sum_{\pi \in \widehat{G}} d_\pi |\widehat{\mathfrak{g}}_\pi|^2 \chi_\pi(e) \\
&= \frac{1}{N} \sum_{\pi \in \widehat{G}} |\widehat{\mathfrak{g}}_\pi|^2 d_\pi^2.
\end{aligned}$$

Since the result is independent of $k$, the optimal frame bounds given by Corollary 4.3.9 are equal and the frame is tight. □

Recall that all the eigenvectors associated to a given representation $\pi \in \widehat{G}$ are associated to the same eigenvalue $\frac{1}{d_\pi} \sum_{g \in S} \chi_\pi(g)$ (see Theorem 4.4.3). However, more than one representation may be associated with the same eigenvalue. Therefore, our assumption in Theorem 4.4.4 is a more relaxed condition than in [103], where $\widehat{\mathfrak{g}}$ is chosen to be constant on each eigenspace for 'good' examples of localization functions.

Applying Theorem 4.4.4 to the translation operators given by (4.15) and (4.18), we immediately obtain the following families of tight frames for Cayley graphs.

COROLLARY 4.4.5. *Let $\Gamma = \mathrm{Cay}(G;S)$ be the Cayley of a finite group $G$ of order $N$ with $S$ closed under conjugation. Assume $\widehat{\mathfrak{g}}$ is constant over every representation of $G$. That is, for every $\pi \in \widehat{G}$,*

$$\widehat{\mathfrak{g}}(\pi_{i,j}) = \widehat{\mathfrak{g}}_\pi$$

*for some constant $\widehat{\mathfrak{g}}_\pi \in \mathbb{C}$ that is independent of $i$ and $j$. Let $\{A_i\}_{i=1}^N$ be either*

1. *The translation operators $T_i\mathfrak{g} = \mathfrak{g} * (\sqrt{N}\delta_i)$ as in Corollary 4.3.12, or*
2. *The repeated shifts $A_i\mathfrak{g} = A_\alpha^{i-1}\mathfrak{g}$ as in Corollary 4.3.13.*

*Then the family of functions $\left\{ \mathfrak{g}_{m,\ell} \;\middle|\; m,\ell = 1,\dots,N \right\}$ defined in (4.8) forms a tight frame with optimal frame bounds $A = B = \sum_{\pi\in\widehat{G}} |\widehat{\mathfrak{g}}_\pi|^2 d_\pi^2$.*

### 4.4.2 Natural Translations on Cayley Graphs

Notice that every Cayley graph comes equipped with its own natural notion of translation, induced by multiplication by a group element. As a consequence, given $g \in G$, we can translate a signal $\mathfrak{f} : G \to \mathbb{C}$ by:

$$\mathfrak{f}(h) \mapsto \mathfrak{f}(g \cdot h), \qquad \text{for } h \in G.$$

Equivalently, the above translation is given by the action of the left (or the right) regular representation:

$$L(g)\mathfrak{f}(h) = \mathfrak{f}(g^{-1}h), \qquad \text{for } h \in G.$$

Our next result shows that the translation operators $T_i$ given by (4.4) are essentially equivalent to the action of the left regular representation for normal Cayley graphs.

Theorem 4.4.6. *Let $\Gamma = \mathrm{Cay}(G;S)$ be the Cayley of a finite group $G$ of order $N$ with $S$ closed under conjugation and equipped with the eigenbasis $\phi_{i,j}^{(k)}$ given by (4.20). Assume $\widehat{\mathfrak{g}}$ is constant over every representation of $G$, i.e., for every $\pi \in \widehat{G}$,*

$$\widehat{\mathfrak{g}}(\pi_{i,j}) = \widehat{\mathfrak{g}}_\pi \in \mathbb{C}.$$

*Then the graph translation operator $T_\ell$ given in (4.4) is given by*

$$(T_\ell \mathfrak{g})(k) = \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \chi_\pi(\ell^{-1}k).$$

*Proof.* We have

$$\begin{aligned}
(T_\ell \mathfrak{g})(v_k) &= \sqrt{N} \sum_{\pi \in \widehat{G}} \sum_{j,i=1}^{d_\pi} \frac{d_\pi}{N} \widehat{\mathfrak{g}}(\pi_{i,j}) \overline{\pi_{i,j}}(\ell) \pi_{i,j}(k) \\
&= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \sum_{j,i=1}^{d_\pi} \overline{\pi_{i,j}}(\ell) \pi_{i,j}(k) \\
&= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \sum_{j,i=1}^{d_\pi} \pi_{j,i}(\ell^{-1}) \pi_{i,j}(k) \\
&= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \sum_{j=1}^{d_\pi} [\pi(\ell^{-1}) \pi(k)]_{j,j} \\
&= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \sum_{j=1}^{d_\pi} [\pi(\ell^{-1}k)]_{j,j} \\
&= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\pi) \chi_\pi(\ell^{-1}k).
\end{aligned}$$

□

Corollary 4.4.7. *Under the conditions of Theorem 4.4.6, the translation operator $T_\ell$ for normal Cayley graphs is invariant when shifted in both indices, that is, for all $m \in G$*

$$(T_\ell \mathfrak{g})(k) = (T_{\ell m} \mathfrak{g})(km) = (T_{m\ell} \mathfrak{g})(mk).$$

*In particular, choosing $m = \ell^{-1}$, we see that*

$$(T_\ell \mathfrak{g})(k) = (T_e \mathfrak{g})(\ell^{-1}k) = L(\ell)[T_e \mathfrak{g}](k),$$

*where $e$ is the group identity element and $L$ is the left regular representation of $G$.*

*Proof.* For the first equality,

$$\begin{aligned}(T_{\ell m}\mathfrak{g})(v_{km}) &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi\big([\ell m]^{-1}km\big) \\ &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi(m^{-1}\ell^{-1}km) \\ &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi(\ell^{-1}k) \\ &= (T_\ell \mathfrak{g})(v_k),\end{aligned}$$

where the last equality follows as characters are class functions. For the second equality,

$$\begin{aligned}(T_{m\ell}\mathfrak{g})(v_{mk}) &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi\big([m\ell]^{-1}mk\big) \\ &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi(\ell^{-1}m^{-1}mk) \\ &= \frac{1}{\sqrt{N}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{\mathfrak{g}}(\phi_\pi)\chi_\pi(\ell^{-1}k) \\ &= (T_\ell \mathfrak{g})(v_k).\end{aligned}$$

□

Notice that Corollary 4.4.7 directly shows that, when translation is given by the operators $T_\ell$, the sharp frame bounds for the associated frame are $A = B = \|T_e \mathfrak{g}\|_2^2$.

Proposition 4.4.8. *Under the assumptions of Theorem 4.4.6*

$$\|T_e \mathfrak{g}\|_2^2 = \sum_{\pi \in \widehat{G}} |\widehat{\mathfrak{g}}_\pi|^2 d_\pi^2,$$

*recovering the expression for the frame bounds given in Corollary 4.4.5.*

*Proof.*

$$
\begin{aligned}
\|T_e\mathfrak{g}\|_2^2 = \sum_{h\in G}|T_e\mathfrak{g}(h)|^2 &= \sum_{h\in G}\left|\frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}} d_\pi\widehat{\mathfrak{g}}(\pi)\chi_\pi(h)\right|^2 \\
&= \frac{1}{N}\sum_{h\in G}\sum_{\pi\in\widehat{G}} d_\pi\widehat{\mathfrak{g}}(\pi)\chi_\pi(h)\sum_{\rho\in\widehat{G}}\overline{d_\rho\widehat{\mathfrak{g}}(\rho)\chi_\rho(h)} \\
&= \frac{1}{N}\sum_{\pi\in\widehat{G}}\sum_{\rho\in\widehat{G}} d_\pi\widehat{\mathfrak{g}}(\pi)\overline{d_\rho\widehat{\mathfrak{g}}(\rho)}\sum_{h\in G}\chi_\pi(h)\overline{\chi_\rho(h)} \\
&= \frac{1}{N}\sum_{\pi\in\widehat{G}}\sum_{\rho\in\widehat{G}} d_\pi\widehat{\mathfrak{g}}(\pi)\overline{d_\rho\widehat{\mathfrak{g}}(\rho)}(N\delta_{\pi,\rho}) \\
&= \sum_{\pi\in\widehat{G}} d_\pi^2|\widehat{\mathfrak{g}}(\pi)|^2 \\
&= \sum_{\pi\in\widehat{G}} |\widehat{\mathfrak{g}}_\pi|^2 d_\pi^2,
\end{aligned}
$$

where we have used Theorem 4.4.6 in the first line and Schur's orthogonality relations (4.19) applied to group characters. □

Additionally, Theorem 4.4.6 and Corollary 4.4.7 show that for $\mathfrak{g}$ defined spectrally, as in [102, 103], to be constant on eigenspaces, the behavior of the translation operator reduces to $T_\ell\mathfrak{g}(k) = L(\ell)T_e\mathfrak{g}(k)$. Then $T_e$ can be viewed as some pre-processing of the localization function $\mathfrak{g}$, which is then translated by the normal definition of group translation. In particular, this shows that with this choice of basis, by Corollary 4.4.5, one can always obtain tight frames for Cayley graphs using translations which respect the graph structure.

A 'natural' choice of translations for graphs would be operators which permute the vertex set while perfectly respecting the structure of the graph. That is, each translation $T$ should satisfy $(Tx, Ty) \in E$ if and only if $(x, y) \in E$. In other words, translation preserves the adjacency structure of $\Gamma$. This is precisely the definition of a *graph automorphism*, and for Cayley graphs, the collection $\{L(g)\}_{g\in G}$ is in fact contained in the collection of all graph automorphisms.

THEOREM 4.4.9. *Let* $\Gamma = \mathrm{Cay}(G; S)$ *for a finite group* $G$ *and any generating set* $S$. *Then the automorphism group of* $\Gamma$ *contains the family* $\{L(g)\}_{g\in G}$ *as a subgroup.*

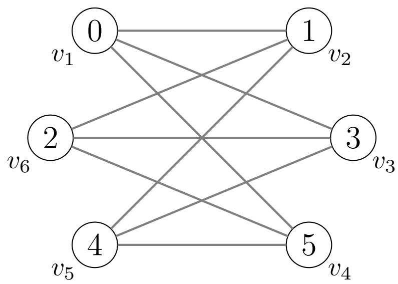
*Proof.* It is sufficient to show that $(x, y)$ is an edge in $\Gamma$ if and only if $(g^{-1}x, g^{-1}y)$ is an edge in $\Gamma$ for every element $g \in G$. However, this is clear as $(x, y)$ is an edge if and only if $S \ni x^{-1}y = x^{-1}gg^{-1}y = (g^{-1}x)^{-1}(g^{-1}y)$, therefore $(g^{-1}x, g^{-1}y)$ is also an edge. Thus $L(g)$ is an automorphism of $\Gamma$. As $L(\cdot)$ is a group homomorphism, it is obvious that its image is a subgroup. □

In Theorem 4.4.9, we do not restrict to normal Cayley graphs. In the special case of Cayley graphs with the generating set closed under conjugation, the collection $\{R(g)\}_{g\in G}$ is also a subgroup of the automorphism group of the graph. The proof is similar, showing $(x, y)$ is an edge if and only if $(xg, yg)$ is an edge, which follows as above and from the additional fact that $S$ is closed under conjugation.

### 4.4.3 Basis of Eigenvectors

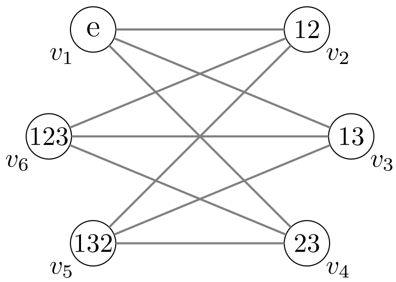
Recall that the eigenvectors of a given graph are not uniquely determined when the graph has repeated eigenvalues, as one must pick a basis for each eigenspace. We now demonstrate how this choice can dramatically impact the properties of the resulting frames.

EXAMPLE 4.4.10. Consider the graph $K_{3,3}$, the complete bipartite graph on 6 vertices with equal partitions. This graph can be realized as the Cayley graph of $\mathbb{Z}_6$ or $S_3$ (Figures 4.1 and 4.2). Depending on the group realization, the eigenvectors chosen from the group representations are considerably different. While it might seem desirable to choose the group to be $\mathbb{Z}_6$, unless we are considering a time series discretization of




$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 & \gamma^5 \\ 1 & \gamma^2 & \gamma^4 & 1 & \gamma^2 & \gamma^4 \\ 1 & \gamma^3 & 1 & \gamma^3 & 1 & \gamma^3 \\ 1 & \gamma^4 & \gamma^2 & 1 & \gamma^4 & \gamma^2 \\ 1 & \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma \end{pmatrix} \end{array} * \frac{1}{\sqrt{6}}$$

**Figure 4.1:** The graph $K_{3,3}$ and basis of coefficient functions of $\mathbb{Z}_6$. $\gamma := \exp[\frac{2\pi i}{6}]$.




$$\begin{array}{c} \\ \text{e} \\ (12) \\ (13) \\ (23) \\ (123) \\ (132) \end{array} \begin{array}{c} \begin{array}{cccccc} \chi_0 & \chi_\pm & \pi_{1,1} & \pi_{2,1} & \pi_{1,2} & \pi_{2,2} \end{array} \\ \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2}\omega^2 & \sqrt{2}\omega & 0 \\ 1 & -1 & 0 & \sqrt{2}\omega & \sqrt{2}\omega^2 & 0 \\ 1 & 1 & \sqrt{2}\omega & 0 & 0 & \sqrt{2}\omega^2 \\ 1 & 1 & \sqrt{2}\omega^2 & 0 & 0 & \sqrt{2}\omega \end{pmatrix} \end{array} * \frac{1}{\sqrt{6}}$$

**Figure 4.2:** The graph $K_{3,3}$ and basis of coefficient functions for $S_3$. $\omega := \exp[\frac{2\pi i}{3}]$.

signals on the real line, the underlying reality of our graph is unlikely to be well-modeled by an abelian group, so it seems unlikely to capture the desired behavior of our irregular domain.

One consequence of these different eigenbases is that for non-isometric translation operators such as the one given in (4.4), the frame bounds can vary based on the choice of the eigenbasis for a fixed graph and fixed window function. Recall that translation in this case can be written as

$$T_j\mathfrak{f} := \sqrt{N}\Phi(\Phi^*\mathfrak{f} \circ \Phi^*\delta_j),$$

where this form makes explicit the dependence on the chosen eigenbasis $\Phi$.

Then the matrix of characters described for $\mathbb{Z}_6$ (Figure 4.1) diagonalizes the graph adjacency matrix (or Laplacian). As all the entries lie in the $\frac{1}{\sqrt{N}}$-radius circle, translation is an isometry leading to a tight frame for any window function. However, another basis can be chosen as this is the graph $\operatorname{Cay}\big(S_3; \big\{(12), (13), (23)\big\}\big)$ (Figure 4.2). Then the coefficient functions of the unitary, irreducible representations of $S_3$ also diagonalize the associated matrices, and in this case the translation operator is not an isometry.

In the basis for $\mathbb{Z}_6$, any non-zero function $\mathfrak{g}$ will provide a tight frame, meaning the ratio between frame bounds $\frac{B}{A} = 1$. However, taking $\mathfrak{g} := \frac{1}{7}(6, 3, 2, 0, 0, 0)^\top$, in the basis provided in Figure 4.2, the ratio is $\frac{69}{29} \approx 2.4$. We remark that any $\mathfrak{g}$ such that $\widehat{\mathfrak{g}}$ is not constant on representations should yield similar examples where the frame is tight in the first basis and not in the second. This provides a stark reminder that one should be careful when choosing an eigenbasis for a given graph.

Chapter 5

# PART II: FRAMES FOR $L^2\left(\mathbb{R}^{n^2}\right)$

## 5.1 Introduction

Our contribution in this chapter is threefold. First, we generalize a 'tiling system' for $L^2(\mathrm{M}_2(\mathbb{R}))$ originally constructed in [58] to $L^2(\mathrm{M}_n(\mathbb{R}))$, for any $n \in \mathbb{N}$. Combining this careful geometric investigation with the theory of square-integrable representations (following the work of [11]) leads to a discretization of the continuous wavelet transform for $L^2(\mathrm{M}_n(\mathbb{R}))$ for any dimension $n$. Through a natural identification between $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, these are also frames for $L^2(\mathbb{R}^{n^2})$.

Second, we compute frame bounds based on the tiling system. Due to our carefully chosen definitions, which differ slightly from [58], we obtain a significant improvement in the frame condition number for the $n = 2$ case. We also find general frame bounds as a function of the dimension $n$. Additionally, we provide an explicit description of the frame in the case $n = 2$.

Third, we find upper bounds on the measure of certain intersections which arise in our tiling system. We use this measure bound to find improved frame bounds for certain classes of signals, leading to frames which are not tight, but for which the frame condition number can be made arbitrarily close to 1. We conclude the section with a discussion of possible extensions to other function spaces.

**Previous Discretizations of the CWT.** Generalized versions of continuous wavelet transforms and frames have been investigated extensively in the past few decades. In the present chapter, we restrict ourselves to constructions on Hilbert spaces, but information on continuous frames in certain Banach spaces can be found in [43]. For examples of wavelet transforms based on the theory of square-integrable representations see [4, 27, 30, 65, 85, 89], and for the general theory governing their construction and behavior see [26, 40, 45, 46, 47, 49].

We also compare our work in this chapter with that of Heinlein and his collaborators in [69, 70]. The two approaches rely on the same representation theoretic viewpoint. Indeed, we both discretize continuous wavelet frames obtained from a square-integrable representation of the affine group; however, we note three key differences. First, their strategy is to use integrated wavelets, which are averages of a wavelet in the Fourier domain over any countable partition of the space (in our case $\mathbb{R}^{n^2}$) into compact sets. With this method, they obtain tight frames, but this comes at the cost that the analysis and reconstruction filters for $n$ resolution scales require $(n+1)$ Fourier transforms. Consequently, the computational cost of this method on higher-dimensional data would be extremely high, whereas our method only requires one Fourier inversion of the analyzing wavelet. Second, we obtain and work with a much more structured decomposition than their general partition (which they call detail decomposition). Our approach starts with a "tile" which covers the space under the action of a discrete, countable set. The structure of our tiling system reduces the amount of information needed for implementation, as one only needs to know the "mother tile" and the form of the discrete set acting on it. Thus, the challenge here is to obtain such suitable tiling systems, which we overcome by carefully investigating the geometry of the space. Third, integrated wavelets require a two-step discretization. That is, they must discretize translations and dilations separately, and it is the intermediate discretization on dilations alone which provides a tight frame. Our method performs the discretization simultaneously, allowing for a one-step process.

**Organization.** The remainder of this chapter is organized as follows. In Section 5.2 we collate all necessary definitions, notations, and background. In Section 5.3, we provide the general theory for constructing discrete frames from the continuous wavelet transform by means of tiling systems. In Section 5.4, we provide a general construction for tiling of $\mathrm{GL}_n(\mathbb{R})$ for any $n \in \mathbb{N}$. In Section 5.5, we compute the frame bounds for the tiling system and derive an upper bound on the frame condition number as a function of the dimension $n$. In Section 5.6, we provide explicit details of the concrete construction for the case when $n = 2$. In Section 5.7, we find a description for signals for which an improved frame bound holds based on the measure of certain intersecting sets arising in the tiling construction. We conclude this chapter by discussing possible extensions of the frame construction to other function spaces.

Portions of this chapter represents joint work with Mahya Ghandehari and Samuel Cogar. The results from Sections 5.3-5.5 can be found in [57] and represents equal work of both authors. The results found in Section 5.7 are mostly my own, and will be found in [72]. Finally, the results for Sobolev spaces in Section 5.8 is joint with Cogar, which we are actively working on improving in [24].

## 5.2 Notations and Definitions

Here we consider the affine group introduced in Example 2.1.7. Recall this is the group $G_n = \mathrm{M}_n(\mathbb{R}) \rtimes_\alpha \mathrm{GL}_n(\mathbb{R})$ with group operation

$$[x_1, h_1][x_2, h_2] = [x_1 + \alpha(h_1)(x_2), h_1 h_2] = [x_1 + h_1 x_2, h_1 h_2],$$

where the action of $h_1 \in \mathrm{GL}_n(\mathbb{R})$ on $x_2 \in \mathrm{M}_n(\mathbb{R})$ is defined to be matrix multiplication: $\alpha(h_1)x_2 = h_1 x_2$. This group can also be seen as affine transformations on $\mathrm{M}_n(\mathbb{R})$ by

$$[x, h]y = hy + x, \qquad \text{for } y \in \mathrm{M}_n(\mathbb{R}).$$

Haar integration on $G_n$ is given by

$$\int_{G_n} f\big([x, h]\big)\, d[x, h] = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} f\big([x, h]\big) \frac{dx\, dh}{|\det(h)|^{2n}},$$

where $dx$ and $dh$ are Lebesgue measure on $\mathbb{R}^{n^2}$ and $f : G_n \to \mathbb{C}$ is compactly supported.

**Transferring to Fourier Domain.** The existence of a continuous wavelet transform depends crucially on the geometric features of the underlying group, and in particular the geometry of the action of $\mathrm{GL}_n(\mathbb{R})$ on the Pontryagin dual of $\mathrm{M}_n(\mathbb{R})$, *i.e.*, the dual group of characters $\widehat{\mathrm{M}_n(\mathbb{R})}$. We now briefly recall and extend the material from Chapter 2 to this setting. To make notation more clear, $\mathrm{M}_n(\mathbb{R})$ is denoted by $A$, when it is identified with $\mathbb{R}^{n^2}$ as an abelian group (see Remark 5.3.3). Let $\widehat{A}$ denote the dual of $\mathrm{M}_n(\mathbb{R})$ under this identification. For the purpose of Fourier analysis on $\mathbb{R}^{n^2}$, identified with $A$, recall that we pair $A$ with $\widehat{A}$ in the following way: for $b = \big(b_{i,j}\big)_{i,j=1}^{n} \in A$, define $\chi_b \in \widehat{A}$ by

$$\chi_b(x) = e^{2\pi i\,\mathrm{Tr}(bx)}, \qquad \text{for } x \in A. \tag{5.1}$$

We have, $\widehat{A} = \big\{\, \chi_b \;\big|\; b \in A \,\big\}$. Thus, $\widehat{A}$ can also be identified with $\mathbb{R}^{n^2}$, and Haar integration on $\widehat{A}$ is simply the Lebesgue integral, *i.e.*,

$$\int_{\widehat{A}} g(\chi)\, d\chi = \int_{\mathbb{R}^{n^2}} g(\chi_b)\, db = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(\chi_{(b_{1,1},\ldots,b_{n,n})})\, db_{1,1} \cdots db_{n,n}. \tag{5.2}$$

For $f \in L^1(A)$, the Fourier transform $\widehat{f}\colon \widehat{A} \to \mathbb{C}$ is given by $\widehat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\, dx$, for all $\chi \in \widehat{A}$. By Theorem 2.2.2, if $f \in L^1(A) \cap L^2(A)$, then $\widehat{f} \in L^2(\widehat{A})$ and $\|\widehat{f}\|_2 = \|f\|_2$. So, the Fourier transform extends to a unitary map $\mathcal{P}\colon L^2(A) \to L^2(\widehat{A})$, the Plancherel transform, such that $\mathcal{P}f = \widehat{f}$, for all $f \in L^1(A) \cap L^2(A)$.

The group $\mathrm{GL}_n(\mathbb{R})$ acts on $A$ by matrix multiplication. This action determines an action of $\mathrm{GL}_n(\mathbb{R})$ on the dual space.

Proposition 5.2.1. *For $b \in A$ and $h \in \mathrm{GL}_n(\mathbb{R})$,*

$$h \cdot \chi_b = \chi_{bh^{-1}} \tag{5.3}$$

*defines a left group action of $\mathrm{GL}_n(\mathbb{R})$ on $\widehat{A}$. Moreover, this action scales Lebesgue measure, so that, for any integrable function $\xi$ on $\widehat{A}$,*

$$\int_{\widehat{A}} \xi(\chi)\, d\chi = |\det(h)|^{-n} \int_{\widehat{A}} \xi(h \cdot \chi)\, d\chi. \tag{5.4}$$

*Proof.* Let $h, k \in \mathrm{GL}_n(\mathbb{R})$ and $b \in \mathrm{M}_n(\mathbb{R})$. Clearly, $I_n \cdot \chi_b = \chi_b$, so to see that (5.3) is an action we calculate

$$k \cdot (h \cdot \chi_b) = k \cdot \chi_{bh^{-1}} = \chi_{bh^{-1}k^{-1}} = \chi_{b(kh)^{-1}} = (kh) \cdot \chi_b.$$

To see that this action scales Lebesgue measure, by (5.2) and (2.1) we have

$$\begin{aligned}
\int_{\widehat{A}} \xi(h \cdot \chi)\, d\chi &= \int\limits_{\mathrm{M}_n(\mathbb{R})} \xi(h \cdot \chi_b)\, db = \int\limits_{\mathrm{M}_n(\mathbb{R})} \xi(\chi_{bh^{-1}})\, db \\
&= \frac{1}{\Delta_{\mathrm{M}_n(\mathbb{R})}(h^{-1})} \int\limits_{\mathrm{M}_n(\mathbb{R})} \xi(\chi_b)\, db \\
&= \Delta_{\mathrm{M}_n(\mathbb{R})}(h) \int_{\widehat{A}} \xi(\chi)\, d\chi
\end{aligned}$$

It follows easily from the calculation in Proposition 2.1.11 that $\Delta_{\mathrm{M}_n(\mathbb{R})}(h) = |\det h|^n$, completing the proof. □

**A Square-Integrable Irreducible Representation.** We now give two equivalent forms of the quasi-regular representation of $G_n$, which is the irreducible representation that has been used in [58] to construct continuous wavelet transforms.

Let $\mathcal{H}$ be a Hilbert space. An irreducible representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of a locally compact group $G$ is called *square-integrable* if there exist nonzero $\eta, \xi \in \mathcal{H}$ such that $\pi_{\eta,\xi} \in L^2(G)$. Such a vector $\eta$ is called *admissible.*

PROPOSITION 5.2.2. *Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be an irreducible square-integrable representation for a locally compact group $G$. Then the set of admissible vectors is dense in $\mathcal{H}$.*

*Proof.* Let $\pi$ and $G$ be as above. Then there exists two non-zero vectors $\eta_0, \xi_0 \in \mathcal{H}$ such that $\pi_{\eta_0,\xi_0} \in L^2(G)$. Define the linear subspace

$$D = \left\{ \eta \in \mathcal{H} \;\middle|\; \pi_{\eta,\xi_0} \in L^2(G) \right\}.$$

Clearly $D$ is non-empty, and for $x \in G$, $\alpha, \beta \in \mathbb{C}$ and $\eta, \gamma \in D$ we have

$$\pi_{\alpha\eta+\beta\gamma,\xi_0}(x) = \langle\, \pi(x)\xi_0, \alpha\eta + \beta\gamma \,\rangle = \alpha\pi_{\eta,\xi_0}(x) + \beta\pi_{\gamma,\xi_0}(x),$$

It follows that $\alpha\eta + \beta\gamma \in D$.

To see this subspace is dense, note that $D$ is $\pi$-invariant as for any $x, y \in G$,

$$\pi_{\pi(y)\eta,\xi_0}(x) = \langle\, \pi(x)\xi_0, \pi(y)\eta \,\rangle = \langle\, \pi(y^{-1}x)\xi_0, \eta \,\rangle = \pi_{\eta,\xi_0}(y^{-1}x).$$

As the Haar measure on $G$ is left-shift invariant, it follows that $\pi(y)\eta \in D$. As $\pi$ is irreducible, we have

$$\overline{\{\, \eta \in \mathcal{H} \mid \pi_{\eta,\xi_0} \in L^2(G) \,\}} = \mathcal{H}. \qquad \square$$

In the present chapter, we will use the following two equivalent, irreducible, square-integrable representations of the affine group to construct a continuous wavelet transform. Notice that (5.5) is exactly the induced representation we obtained in Section 2.1.3.3.

Proposition 5.2.3. *Let $\mathcal{H}_1 = L^2(A)$ and $\mathcal{H}_2 = L^2(\widehat{A})$. Then,*

*(i) $\rho$ is a unitary representation, where*

$$\rho : G_n \to \mathcal{U}(\mathcal{H}_1), \qquad \big(\rho[x,h]g\big)(y) = |\det(h)|^{-n/2} g\big(h^{-1}(y - x)\big), \tag{5.5}$$

*for all $[x,h] \in G_n$ and $y \in A$.*

*(ii) $\pi$ is a unitary representation, where*

$$\pi : G_n \to \mathcal{U}(\mathcal{H}_2), \qquad \big(\pi[x,h]\xi\big)(\chi) = |\det(h)|^{n/2}\, \overline{\chi(x)}\, \xi(h^{-1} \cdot \chi),$$

*for all $[x,h] \in G_n$ and $\chi \in \widehat{A}$.*

*(iii) The representations $\rho$ and $\pi$ are equivalent square-integrable irreducible representations of $G_n$. Namely, we have $\pi[x,h]\mathcal{P} = \mathcal{P}\rho[x,h]$ for all $[x,h] \in G_n$, where $\mathcal{P}$ is the Plancherel transform.*

*Proof.* Parts (i) and (ii) are proven in [58, Proposition 4.2], so we omit the proof. For part (iii), let $f \in C_c(\mathrm{M}_n(\mathbb{R}))$. We now use the results of Proposition 2.2.6, noting that $\rho[x,h]f = (T_x D_h f)$, to compute

$$\begin{aligned}
[\mathcal{P}\rho[x,h]f](\chi) &= (\widehat{T_x D_h f})(\chi) \\
&= \overline{\chi(x)}(\widehat{D_h f})(\chi) \\
&= \overline{\chi(x)}|\det h|^{n/2}\widehat{f}(h^{-1}\cdot\chi) \\
&= \big[\pi\big([x,h]\big)\mathcal{P}f\big](\chi).
\end{aligned}$$

As $C_c\big(\mathrm{M}_n(\mathbb{R})\big)$ is dense in $L^2(G)$, the unitary operator $\mathcal{P}$ intertwines $\pi$ and $\rho$, thus the two representations are equivalent. □

**Continuous Wavelet Transforms.** We now use the square-integrable representations defined in the previous section to construct a continuous wavelet transform. Let $\rho$ be the square-integrable representation defined in (5.5).

DEFINITION 5.2.4. A function $\psi \in L^2(A)$ is called a *wavelet* if

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} \left|\,\widehat{\psi}(\chi_h)\,\right|^2 \frac{dh}{|\det h|^n} = 1. \tag{5.6}$$

A *continuous wavelet transform* (CWT) associated with a function $\psi \in L^2(A)$ is the linear transformation defined as

$$V_\psi\colon L^2(\mathrm{M}_n(\mathbb{R})) \to L^2(G_n), \qquad V_\psi f[x,h] = \langle\, f, \rho[x,h]\psi\,\rangle, \tag{5.7}$$

for $f \in L^2(A), [x,h] \in G_n$.

PROPOSITION 5.2.5 (cf. [45, Theorem 2.25]). *Let $V_\psi$ be the CWT in (5.7). Then $V_\psi$ is an isometry if and only if $\psi$ is a wavelet.*

*Proof.* Let $\psi$ be a function, then for $g \in L^2(A)$ we have, by Plancherel's theorem (Theorem 2.2.2),

$$\|v_\psi g\|^2_{L^2(G_n)} = \int_{G_n} |\langle\, g, \rho[x,h]\psi\,\rangle_{L^2(A)}|^2 d[x,h]$$

$$= \int_{G_n} |\langle\, \widehat{g}, \pi[x,h]\widehat{\psi}\,\rangle_{L^2(\widehat{A})}|^2 d[x,h].$$

Calculating the inner product shows that

$$\|v_\psi g\|^2_{L^2(G_n)} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} \left| \int\limits_{\mathrm{M}_n(\mathbb{R})} \widehat{g}(\chi_b) |\det h|^{n/2} \chi_b(x) \overline{\widehat{\psi}(\chi_{bh^{-1}})} db \right|^2 \frac{dx\, dh}{|\det h|^{2n}}$$

$$= \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{\phi}_h(-x)|^2 \frac{dx\, dh}{|\det h|^n},$$

where we have defined the new function $\phi_h(b) = \widehat{g}(\chi_b)\overline{\widehat{\psi}(\chi_{bh})}$. We are now in a position to apply Plancherel's theorem again to see that

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} \left| \widehat{\phi}_h(-x) \right|^2 \frac{dx\, dh}{|\det h|^n} = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{g}(\chi_b)\overline{\widehat{\psi}(\chi_{bh})}|^2 \frac{db\, dh}{|\det h|^n}$$

$$= \int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{g}(\chi_b)|^2 \left( \int\limits_{\mathrm{GL}_n(\mathbb{R})} |\widehat{\psi}(\chi_{bh})|^2 \frac{dh}{|\det h|^n} \right) db$$

Using Proposition 2.1.11, we see

$$\int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{g}(\chi_b)|^2 \left( \int\limits_{\mathrm{GL}_n(\mathbb{R})} \left|\widehat{\psi}(\chi_{bh})\right|^2 \frac{dh}{|\det h|^n} \right) db = \int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{g}(\chi_b)|^2 \left( \int\limits_{\mathrm{GL}_n(\mathbb{R})} \left|\widehat{\psi}(\chi_h)\right|^2 \frac{dh}{|\det h|^n} \right) db$$

$$= \int\limits_{\mathrm{M}_n(\mathbb{R})} |\widehat{g}(\chi_b)|^2 \, db \cdot \left( \int\limits_{\mathrm{GL}_n(\mathbb{R})} \left|\widehat{\psi}(\chi_h)\right|^2 \frac{dh}{|\det h|^n} \right)$$

$$= \|g\|^2 \cdot \left( \int\limits_{\mathrm{GL}_n(\mathbb{R})} \left|\widehat{\psi}(\chi_h)\right|^2 \frac{dh}{|\det h|^n} \right).$$

Therefore $V_\psi$ is an isometry if and only if $\psi$ satisfies (5.6). □

In particular, Proposition 5.2.5 implies that for any $f, g \in L^2(A)$

$$\begin{aligned} \langle f, g \rangle_{L^2(A)} &= \langle V_\psi f, V_\psi g \rangle_{L^2(G_n)} \\ &= \int_{G_n} \langle f, \psi^{x,h} \rangle_{L^2(A)} \langle \psi^{x,h}, g \rangle_{L^2(A)} \, d[x,h] \\ &= \left\langle \int_{G_n} \langle f, \psi^{x,h} \rangle_{L^2(A)} \psi^{x,h} \, d[x,h], g \right\rangle_{L^2(A)}, \end{aligned}$$

where we denote $\rho[x,h]\psi$ by $\psi^{x,h}$. This formula can be written in the following weak integral form:

$$f = \int_{G_n} \langle f, \rho[x,h]\psi \rangle \, \rho[x,h]\psi \, d[x,h], \tag{5.8}$$

for any $f \in L^2(A)$.

For a comprehensive discussion of continuous wavelet transforms, and their connection with square-integrable representations, see [45].

## 5.3 Constructing Frames using CWTs

In this section, we present a summary of the results from [11], explaining how CWTs may be used to construct discrete frames. The objective here is to construct frames of the form $\left\{\rho[x,h]\psi\right\}_{[x,h]\in P}$, where $P$ is a discrete subset of $G_n$, and $\rho$ and $\psi$ are as in Definition 5.2.4. The frame constructions in [11] heavily rely on the concept of a 'tiling system' which we define here.

In [58], Ghandehari, Syzdykova, and Taylor use the approach of Bernier and Taylor ([11]) to construct a discrete frame for $L^2(\mathrm{M}_2(\mathbb{R}))$. More precisely, they obtain a suitable tiling system for $\mathrm{GL}_2(\mathbb{R})$, which they use to discretize the continuous wavelet transform on $\mathbb{R}^4$. In this section, we extend their frame construction to obtain discrete frames for $L^2(\mathrm{M}_n(\mathbb{R}))$, for any $n \geq 2$. In addition, we obtain a much tighter gap between the frame bounds.

In what follows, we restrict ourselves to the groups which are being studied in this chapter: $\mathrm{M}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})$, and the affine group $G_n$; however, our methods can be carried out in more general settings.

Definition 5.3.1. Let $P$ be a countable subset of $\mathrm{GL}_n(\mathbb{R})$, and $F$ be an open relatively compact subset of $\mathrm{GL}_n(\mathbb{R})$. The pair $(F,P)$ is called a *tiling system* for $\mathrm{GL}_n(\mathbb{R})$ if the following two conditions are satisfied:

(i) $\lambda_{\mathrm{GL}_n}\left(\overline{F}\cdot p \bigcap \overline{F}\cdot q\right) = 0$ for every distinct pair $p,q\in P$,

(ii) $\lambda_{\mathrm{GL}_n}\left(\mathrm{GL}_n(\mathbb{R}) \setminus \bigcup_{p\in P}\overline{F}\cdot p\right) = 0$,

where $\lambda_{\mathrm{GL}_n}$ denotes the left Haar measure of $\mathrm{GL}_n(\mathbb{R})$, and $\overline{F}$ denotes the closure of $F$ in $\mathrm{GL}_n(\mathbb{R})$. Note that conditions (i) and (ii) remain unchanged if one replaces $\lambda_{\mathrm{GL}_n}$ with $\lambda_{\mathrm{M}_n}$, the left Haar measure on $\mathrm{M}_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$.

Remark 5.3.2. In the above definition, we think of a subset $S$ of $A = \mathrm{M}_n(\mathbb{R})$ as a subset $\widetilde{S}$ of $\widehat{A}$ in the following natural manner: $\widetilde{S} = \left\{\chi_b \;\middle|\; b\in S\right\}$. With this identification in mind, we have $\widetilde{\overline{F}\cdot p} = p^{-1}\cdot\widetilde{\overline{F}}$ where the left group action on $\widehat{A}$ is given by (5.3).

Remark 5.3.3. Recall that we identify elements of $\mathrm{M}_n(\mathbb{R})$ with vectors in $\mathbb{R}^{n^2}$ via the map $\Phi$ given by

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix} \overset{\Phi}{\longmapsto} \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & \dots & x_{n,1} & \dots & x_{n,n} \end{pmatrix}^{\top}. \tag{5.9}$$

The $n \times n$ matrix structure is only used when multiplication is involved. This identification allows us to transfer the notion of tiling to $\mathbb{R}^{n^2}$. Let $(F, P)$ be a tiling system for $\mathrm{GL}_n(\mathbb{R})$. For each $p \in P$, let $L^2\big(\Phi(\overline{F} \cdot p)\big)$ denote the closed subspace of $L^2(\mathbb{R}^{n^2})$ consisting of functions that are zero almost everywhere on $\mathbb{R}^{n^2} \setminus \Phi(\overline{F} \cdot p)$. Noting that $\mathrm{GL}_n(\mathbb{R})$ is a co-null subset of $\mathrm{M}_n(\mathbb{R})$ (and thus $\mathbb{R}^{n^2}$), we have that

$$L^2\left(\mathbb{R}^{n^2}\right) = \bigoplus_{p \in P} L^2\big(\Phi(\overline{F} \cdot p)\big).$$

With this in mind, we can think of $(F, P)$ as a tiling system for $\mathrm{M}_n(\mathbb{R})$, or equivalently for $\mathbb{R}^{n^2}$, as well.

We now give a brief review of the method introduced in [11] for constructing a frame from a tiling system. We first introduce some notations, which will be useful for the next theorem.

Notation 5.3.4. Let $(F, P)$ be a tiling system, and let

$$R = \left\{ [x_{i,j}] \in \mathrm{M}_n(\mathbb{R}) \;\middle|\; a_{i,j} \leq x_{i,j} \leq b_{i,j} \;\text{ for }\; i, j = 1, \dots, n \right\}$$

be a hypercube in $\mathrm{M}_n(\mathbb{R})$ whose interior contains $\overline{F}$. Let $|R| = \displaystyle\prod_{i,j=1}^{n} (b_{i,j} - a_{i,j})$ be the volume of $R$, and

$$J = \left\{ \left[ \frac{m_{i,j}}{b_{j,i} - a_{j,i}} \right]_{i,j} \in \mathrm{M}_n(\mathbb{R}) \;\middle|\; m_{i,j} \in \mathbb{Z} \right\}.$$

Every $\gamma \in J$ defines a character on the hypercube $R \subseteq \mathrm{M}_n(\mathbb{R})$ as follows:

$$e_\gamma(y) = \frac{1}{\sqrt{|R|}} \mathbb{1}_R(y)\, e^{2\pi i \operatorname{Tr}(\gamma y)}, \qquad \text{for } y \in R,$$

where $\mathbb{1}_R$ is the characteristic function of $R$. Then $\left\{ e_\gamma \;\middle|\; \gamma \in J \right\}$ is an orthonormal basis for $L^2(R)$.

We now prove a slightly different version of Theorem 3 of [11]. Even though the proofs are similar, we manage to obtain a much tighter frame condition number due to our new definition for the constant $M$. In Section 5.4, we will show that the conditions in our theorem can be met for $G_n = \mathrm{M}_n(\mathbb{R}) \rtimes \mathrm{GL}_n(\mathbb{R})$ for any $n \in \mathbb{N}$.

Notation 5.3.5. Let $(F, P)$ be a tiling system for $\mathrm{GL}_n(\mathbb{R})$, and $R$ be a hypercube containing $\overline{F}$. Let $F_\mathrm{o}$ be an open set satisfying $\overline{F} \subseteq F_\mathrm{o} \subseteq R$. For $p \in P$, define $I_{F_\mathrm{o}}(p) := \left\{ k \in P \;\middle|\; \overline{F} \cdot p \cap F_\mathrm{o} \cdot k \neq \emptyset \right\}$. Let

$$M := \sup_{p \in P} |I_{F_\mathrm{o}}(p)|.$$

Observe that $M$ is finite according to Lemma 4 of [11].

Theorem 5.3.6. *Let $(F, P)$ be a tiling system for $\mathrm{GL}_n(\mathbb{R})$, with $R$, $F_\mathrm{o}$, and $M$ be as in Notation 5.3.5. Let $g \in L^2(\mathrm{M}_n(\mathbb{R}))$ be such that $\mathbb{1}_{\overline{F}} \leq \widehat{g} \leq \mathbb{1}_{F_\mathrm{o}}$. Then the collection of vectors $\left\{ \rho[\lambda, p]^{-1} g \;\middle|\; (\lambda, p) \in J \times P \right\}$ is a discrete frame in $L^2(\mathrm{M}_n(\mathbb{R}))$, where $J$ is defined in Notation 5.3.4. Moreover, lower and upper frame bounds are given by*

$$|R| \, \|f\|_2^2 \leq \sum_{k \in P} \sum_{\gamma \in J} \left| \langle f, \rho[\gamma, k]^{-1} g \rangle_{L^2(\mathrm{M}_n(\mathbb{R}))} \right|^2 \leq M |R| \, \|f\|_2^2.$$

*Proof.* For an arbitrary $f \in L^2(\mathrm{M}_n(\mathbb{R}))$, we have by Proposition 5.2.3 that

$$\begin{aligned}
\sum_{k \in P} \sum_{\gamma \in J} \left| \langle f, \rho[\gamma, k]^{-1} g \rangle_{L^2(A)} \right|^2 &= \sum_{k \in P} \sum_{\gamma \in J} \left| \langle \pi[\gamma, k] \widehat{f}, \widehat{g} \rangle_{L^2(\widehat{A})} \right|^2 \\
&= \sum_{k \in P} \sum_{\gamma \in J} \left| \int_A (\pi[\gamma, k] \widehat{f})(\chi_b) \overline{\widehat{g}(\chi_b)} \, db \right|^2 \\
&= \sum_{k \in P} \sum_{\gamma \in J} \left| \int_A |\det k|^{n/2} \overline{\chi_b(\gamma)} \widehat{f}(\chi_{bk}) \overline{\widehat{g}(\chi_b)} \, db \right|^2 \\
&= \sum_{k \in P} \sum_{\gamma \in J} \left| \int_R |\det k|^{n/2} \overline{\chi_b(\gamma)} \widehat{f}(\chi_{bk}) \overline{\widehat{g}(\chi_b)} \, db \right|^2,
\end{aligned}$$

where the last equality follows as $\widehat{g} = 0$ outside $R$. We proceed by defining the new function $h_k(b) = |\det k|^{n/2}\, \widehat{f}(\chi_{bk})\, \overline{\widehat{g}(\chi_b)}$. Then note $\left\{ e_\gamma(b) = \chi_b(\gamma) \mathbb{1}_R(b) \frac{1}{\sqrt{|R|}} \;\middle|\; \gamma \in J \right\}$ forms an orthonormal basis for $R$. Then we are in a position to apply Parseval's identity on $L^2(R)$ to see that

$$\begin{aligned}
\sum_{k\in P}\sum_{\gamma\in J} \left| \langle f, \rho[\gamma,k]^{-1} g \rangle_{L^2(A)} \right|^2 &= |R| \sum_{k\in P}\sum_{\gamma\in J} \left| \langle h_k, e_\gamma \rangle_{L^2(R)} \right|^2 \\
&= |R| \sum_{k\in P} \int_R |\det k|^n\, |\widehat{f}(\chi_{bk})|^2\, |\widehat{g}(\chi_b)|^2 \, db \\
&= |R| \sum_{k\in P} \int_A |\widehat{f}(\chi_b)|^2\, |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \\
&= |R| \int_A |\widehat{f}(\chi_b)|^2 \left( \sum_{k\in P} |\widehat{g}(\chi_{bk^{-1}})|^2 \right) db.
\end{aligned}$$

Note that for almost every $b \in \mathrm{M}_n(\mathbb{R})$, there exists an element $k \in P$ for which $b \in \overline{F} \cdot k$. This, together with the fact that $\widehat{g} \geq \mathbb{1}_{\overline{F}}$, implies that $\sum_{k\in P} |\widehat{g}(\chi_{bk^{-1}})|^2 \geq 1$ for almost every $b \in \mathrm{M}_n(\mathbb{R})$. Thus, $|R|$ is a lower bound for the frame. To obtain the upper bound, we note that $\widehat{g} \leq 1$. Moreover, for almost every $b \in \mathrm{M}_n(\mathbb{R})$, there are at most $M$ values of $k \in P$ for which $|\widehat{g}(\chi_{bk^{-1}})| > 0$, as $\widehat{g} \leq \mathbb{1}_{F_\circ}$. This completes the proof. □

## 5.4 Tiling System for $\mathrm{GL}_n(\mathbb{R})$

In this section, we generalize the construction of a tiling system for $\mathrm{GL}_2(\mathbb{R})$ given in [58] to a tiling system for $\mathrm{GL}_n(\mathbb{R})$. We then show this construction meets the conditions of Theorem 5.3.6. We compute the corresponding frame bounds in Section 5.5.

The definition of our tiling system in this section and the computations thereafter are inspired by the Iwasawa decomposition for $\mathrm{GL}_n(\mathbb{R})$. For matrix groups, this is equivalent to the well-known Gram decomposition (*i.e.*, the $QR$ decomposition), with the upper triangular part further decomposed into a diagonal matrix with positive entries and a unit upper triangular matrix (*i.e.*, the $LU$ factorization of $R$). Finally, we factor out the determinant to use as a parameter in our tiling system. While this realization works for the current setting, the Iwasawa decomposition is much more general, and could be applied to other semisimple Lie groups. This decomposition can be viewed as a change of variables when computing the Haar measure, for which we provide the relevant formulas here as well.

THEOREM 5.4.1 ([76, Proposition 2.3]). *Each $a \in \mathrm{GL}_n(\mathbb{R})$ can be uniquely decomposed as $a = skwy$, where $s \in \mathbb{R}^+$, $k \in \mathrm{O}_n$, $w \in \mathrm{D}_n$, and $y \in \mathrm{T}_n$. Here $\mathrm{O}_n$ is the orthogonal group in dimension $n$, $\mathrm{D}_n$ is the group of diagonal matrices with positive diagonal entries and determinant 1, and $\mathrm{T}_n$ is the group of unit upper triangular matrices. That is*

$$\mathrm{GL}_n(\mathbb{R}) = \left\{ sk \begin{pmatrix} w_1 & & & \\ & \ddots & & \\ & & w_{n-1} & \\ & & & \prod_{i=1}^{n-1} w_i^{-1} \end{pmatrix} \begin{pmatrix} 1 & y_{1,2} & \cdots & y_{1,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & y_{n-1,n} \\ & & & 1 \end{pmatrix} \;\middle|\; \begin{array}{r} s, w_i \in \mathbb{R}^+, \\ y_{i,j} \in \mathbb{R}, \\ k \in \mathrm{O}_n \end{array} \right\}.$$

*This is known as the Iwasawa decomposition. Moreover, the Haar measure of a subset $E$ of $\mathrm{GL}_n(\mathbb{R})$ can be computed in terms of the Euclidean coordinates in this decomposition as*

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E) = \int\limits_{\mathbb{R}^+ \times \mathrm{O}_n \times \mathrm{D}_n \times \mathrm{T}_n} \mathbb{1}_E(skwy) s^{-1} \prod_{i=1}^{n-1} w_i^{2(n-i)-1} \, dk \, ds \prod_{i=1}^{n-1} dw_i \prod_{i<j} dy_{i,j},$$

*where $dk$ is the normalized Haar measure on $\mathrm{O}_n$, and $ds, dw_i, dy_{i,j}$ are Lebesgue measure on $\mathbb{R}$.*

We include two proofs, one using general theory related to semidirect product groups, the second relying on direct computation of the Jacobian when the decomposition is viewed as a change of variables.

*Proof 1.* In this proof, we denote the Haar measure of a group $G$ by $\mu_G$. First note that $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is a group homomorphism, where $\mathbb{R}^*$ is the multiplicative group of nonzero real numbers. Clearly, $H := \det^{-1}\big(\{1, -1\}\big)$ is a closed normal subgroup of $\mathrm{GL}_n(\mathbb{R})$, and $\mathbb{R}^+$ is isomorphic with $\mathrm{GL}_n(\mathbb{R})/H$. So by [41, Theorem 2.51], the Haar measure of $\mathrm{GL}_n(\mathbb{R})$ can be decomposed as $d\mu_{\mathrm{GL}_n}(sx) = d\mu_{\mathbb{R}^+}(s)\, d\mu_H(x)$. Next, consider the Iwasawa decomposition $\mathrm{SL}_n(\mathbb{R}) = \mathrm{SO}_n\mathrm{D}_n\mathrm{T}_n$, and note that by taking the inverse and allowing matrices with determinant of -1, we can write $H = \mathrm{T}_n\mathrm{D}_n\mathrm{O}_n$. Since $\mathrm{O}_n$ and $H$ are unimodular groups, we can apply [41, Theorem 2.51] again, to get the decomposition $d\mu_H(ywk) = d\mu_{\mathrm{O}_n}(k)\, d\mu_{\mathrm{T}_n\mathrm{D}_n}(yw)$.

Finally, we need to compute $d\mu_{\mathrm{T}_n\mathrm{D}_n}$. To do so, note that

$$(yw)(y'w') = (y(wy'w^{-1}))\,(ww').$$

Viewing both $\mathrm{D}_n$ and $\mathrm{T}_n$ as subgroups of the group of upper triangular matrices with positive determinant, clearly $\mathrm{T}_n$ is a complementary subgroup of $\mathrm{D}_n$, in the sense discussed in Section 2.1.3.2. Consequently, $\mathrm{D}_n$ normalizes $\mathrm{T}_n$ (*i.e.*, $w^{-1}yw \in \mathrm{T}_n$, whenever $w \in \mathrm{D}_n$ and $y \in \mathrm{T}_n$) and we can view $\mathrm{T}_n\mathrm{D}_n$ as a semidirect product $\mathrm{T}_n \rtimes_\beta \mathrm{D}_n$, with $\beta(w)(y) = wyw^{-1}$. Therefore, by standard results in semidirect products of groups (*e.g.* see Section 1.2 of [78]), we have $d\mu_{\mathrm{T}_n\mathrm{D}_n}(yw) = \delta(w)^{-1} d\mu_{\mathrm{T}_n}(y)\, d\mu_{\mathrm{D}_n}(w)$, where $\delta : \mathrm{D}_n \to \mathbb{R}^+$ satisfies

$$\int_{\mathrm{T}_n} f(y) d\mu_{\mathrm{T}_n}(y) = \delta(w) \int_{\mathrm{T}_n} f(wyw^{-1}) dy, \qquad \text{for every } f \in C_c(\mathrm{T}_n) \text{ and } w \in \mathrm{D}_n.$$

One can easily see that $d\mu_{\mathrm{D}_n}(w) = \prod_{i=1}^{n-1} \frac{dw_i}{w_i}$ and $d\mu_{\mathrm{T}_n}(y) = \prod_{i<j} dy_{i,j}$, with matrices $w$ and $y$ as represented in the statement of the theorem. Given that $wyw^{-1} = [y'_{i,j}] \in \mathrm{T}_n$ with $y'_{i,j} = \frac{w_i y_{i,j}}{w_j}$, we have $\delta(w) = \prod_{1\le i<j\le n} \frac{w_j}{w_i}$. We can compute $\delta$ by a simple counting argument, and using the fact that $w_n = (w_1 \cdots w_{n-1})^{-1}$, as follows:

$$\delta(w) = \prod_{1\le i<j\le n} \frac{w_j}{w_i} = \frac{\prod_{1\le i\le n} w_i^{i-1}}{\prod_{1\le i\le n} w_i^{n-i}} = \prod_{1\le i\le n} w_i^{2i-n-1} = \prod_{1\le i\le n-1} w_i^{2i-2n}.$$

Putting all these together, we obtain the Haar measure of $\mathrm{GL}_n(\mathbb{R})$, when the decomposition $\mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^+\mathrm{T}_n\mathrm{D}_n\mathrm{O}_n$ is used:

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E) = \int\limits_{\mathbb{R}^+\times\mathrm{T}_n\times\mathrm{D}_n\times\mathrm{O}_n} \mathbb{1}_E(sywk) \prod_{i=1}^{n-1} w_i^{2i-2n}\, dk\, \frac{ds}{s} \prod_{i=1}^{n-1} \frac{dw_i}{w_i} \prod_{i<j} dy_{i,j}.$$

Finally, applying the inverse map to the above formula and using unimodularity of $\mathrm{GL}_n(\mathbb{R})$, we get

$$\begin{aligned}\lambda_{\mathrm{GL}_n(\mathbb{R})}(E) &= \int\limits_{\mathbb{R}^+\times\mathrm{O}_n\times\mathrm{D}_n\times\mathrm{T}_n} \mathbb{1}_E(s^{-1}k^{-1}w^{-1}y^{-1}) \prod_{i=1}^{n-1} w_i^{2i-2n}\, dk\, \frac{ds}{s} \prod_{i=1}^{n-1} \frac{dw_i}{w_i} \prod_{i<j} dy_{i,j} \\ &= \int\limits_{\mathbb{R}^+\times\mathrm{O}_n\times\mathrm{D}_n\times\mathrm{T}_n} \mathbb{1}_E(skwy) \prod_{i=1}^{n-1} w_i^{-2i+2n}\, dk^{-1}\, \frac{ds^{-1}}{s^{-1}} \prod_{i=1}^{n-1} \frac{dw_i^{-1}}{w_i^{-1}} \prod_{i<j} d(-y_{i,j}) \\ &= \int\limits_{\mathbb{R}^+\times\mathrm{O}_n\times\mathrm{D}_n\times\mathrm{T}_n} \mathbb{1}_E(skwy) \prod_{i=1}^{n-1} w_i^{-2i+2n}\, dk\, \frac{ds}{s} \prod_{i=1}^{n-1} \frac{dw_i}{w_i} \prod_{i<j} dy_{i,j}.\end{aligned}$$

□

We also provide an alternate proof by treating the above matrix decomposition as a change of variables and directly computing the determinant of the Jacobian.

Notation 5.4.2. From this point forward, we use $\mathrm{diag}(w_1, \ldots, w_n)$ to denote an $n \times n$ diagonal matrix with diagonal entries $w_1, \ldots, w_n$.

*Proof 2.* By [36, Proposition 5.3.2], the decomposition of $\mathrm{GL}_n(\mathbb{R})$ with left Haar measure into $kt$ where $k \in \mathrm{O}_n$ and $t$ is upper triangular with positive diagonal entries is given by

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} f(x)\, \frac{dx}{|\det x|^n} = c_n \int\limits_{\mathrm{O}_n\times\mathrm{T}_n} f(kt)\, dk \prod_{i=1}^{n} t_{i,i}^{-i} \prod_{i\leq j} dt_{i,j},$$

where $c_n$ is a constant depending only on the dimension $n$. Let $t = (t_{i,j})_{i\leq j}$ be the upper triangular matrix in the decomposition above and let $J$ denote the Jacobian matrix for the change of variables $t_{i,j} = sw_i y_{i,j}$. Let $k_n = \frac{n(n+1)}{2}$, then $J$ is a $k_n \times k_n$ matrix. For notational convenience, let $w_n = \prod_{i=1}^{n-1} w_i^{-1}$. Then noting $y_{i,i} = 1$ for each $i$, one can compute $|\det J|$ by carefully choosing the ordering of the rows and columns of $J$. Let the rows be indexed by the coordinates of $t$ as (in order)

$$(1,1),(2,2),\cdots(n-1,n-1),(1,2),(1,3),\cdots,(1,n),(2,3),\cdots,(2,n),\cdots(n-1,n),(n,n).$$

We index the columns more naturally by the new variables as

$$s, w_1, w_2, \cdots, w_{n-1}, y_{1,2}, y_{1,3}, \cdots y_{1,n}, y_{2,3}, \cdots, y_{n-1,n}.$$

Now, we work with the following block form of the Jacobian matrix

$$J = \begin{pmatrix} J_{1,1} & J_{1,2} & 0 \\ J_{2,1} & J_{2,2} & J_{2,3} \\ w_n & J_{3,2} & 0 \end{pmatrix},$$

where $J_{1,1}$ is $(n-1)\times 1$, $J_{1,2}$ is $(n-1)\times(n-1)$, $J_{2,1}$ is $(k_n-n)\times 1$, $J_{2,2}$ is $(k_n-n)\times(k_n-n)$, $J_{2,3}$ is $(k_n-n)\times(k_n-n)$, and $J_{3,2}$ is $1\times(n-1)$. It is easy to verify that

$$\begin{aligned}
J_{1,1} &= (w_1, w_2, \cdots, w_{n-1})^\top, \\
J_{1,2} &= \operatorname{diag}(s, s, s, \cdots, s), \\
J_{2,3} &= \operatorname{diag}(sw_1, sw_1, \cdots, sw_1, sw_2, \cdots, sw_2, \cdots, sw_{n-1}), \text{ and} \\
J_{3,2} &= (-sw_1^{-1}w_n, -sw_2^{-1}w_n, \cdots, -sw_{n-1}^{-1}w_n),
\end{aligned}$$

where we omit the entries of $J_{2,1}$ and $J_{2,2}$ as they do not enter the calculation of the determinant. Note that in $J_{2,3}$, each term $sw_i$ appears $n-i$ times (once for each $w_i y_{i,j}$ in row $i$ of $t$).

Let $M_i$ denote the matrix obtained by removing row $k_n$ and column $i$. Then by expanding along the last row, we obtain

$$\det(J) = (-1)^{k_n+1} w_n \det(M_1) + \sum_{i=2}^{n} (-1)^{k_n+i+1} \left(sw_{i-1}^{-1} w_n\right) \det(M_i),$$

where $M_1$ and $M_2$ are lower triangular. Then observe that for $3 \le i \le n$, the matrix $M_i$ can be made lower triangular through $i-2$ row permutations given in cyclic notation by

$(123\dots(i-1))$. This shifts the value $w_{i-1}$ to the entry $(M_i)_{1,1}$ and otherwise leaves the diagonal unchanged from $M_1$. As each permutation introduces a factor of $(-1)^{i-2}$, the sign in the summation becomes $(-1)^{k_n+2i-1} = (-1)^{k_n+1}$. It easily follows that each term in the sum becomes equal to $(-1)^{k_n+1}\det(M_1) = (-1)^{k_n+1} w_{i-1} w_{i-1}^{-1} w_n s^{k_n-1} \prod_{i=1}^{n-1} w_i^{n-i}$. Since the sum contains $n$ copies of this expression, we obtain

$$|\det(J)| = n \cdot s^{\frac{n(n+1)}{2}-1} \cdot \prod_{i=1}^{n-2} w_i^{n-i-1}.$$

Combining the first and second change of variables and appropriately rescaling the Haar measure completes the proof. □

**Construction of Tiling System.** We now show how the tiling system originally constructed for $\mathrm{GL}_2(\mathbb{R})$ in [58] may be extended to a tiling system for $\mathrm{GL}_n(\mathbb{R})$.

Let $F$ be the set

$$F = \left\{ sk \begin{pmatrix} w_1 & w_1 y_{1,2} & w_1 y_{1,3} & \cdots & w_1 y_{1,n-1} & w_1 y_{1,n} \\ & w_2 & w_2 y_{2,3} & \cdots & w_2 y_{2,n-1} & w_2 y_{2,n} \\ & & w_3 & \cdots & w_3 y_{3,n-1} & w_3 y_{3,n} \\ & & & \ddots & & \vdots \\ & & & & w_{n-1} & w_{n-1} y_{n-1,n} \\ & & & & & \prod_{i=1}^{n-1} w_i^{-1} \end{pmatrix} \middle| \begin{array}{r} s, w_i \in [1,2), \\ y_{i,j} \in [0,1), \\ k \in \mathrm{O}_n \end{array} \right\},$$

and let $P$ be the discrete set

$$P = \left\{ 2^{\lambda} \begin{pmatrix} 2^{\kappa_1} & 2^{\kappa_2}\mu_{1,2} & 2^{\kappa_3}\mu_{1,3} & \cdots & 2^{\kappa_{n-1}}\mu_{1,n-1} & 2^{\kappa_n}\mu_{1,n} \\ & 2^{\kappa_2} & 2^{\kappa_3}\mu_{2,3} & \cdots & 2^{\kappa_{n-1}}\mu_{2,n-1} & 2^{\kappa_n}\mu_{2,n} \\ & & 2^{\kappa_3} & \cdots & 2^{\kappa_{n-1}}\mu_{3,n-1} & 2^{\kappa_n}\mu_{3,n} \\ & & & \ddots & & \vdots \\ & & & & 2^{\kappa_{n-1}} & 2^{\kappa_n}\mu_{n-1,n} \\ & & & & & 2^{\kappa_n} \end{pmatrix} \middle| \begin{array}{r} \lambda, \kappa_i, \mu_{i,j} \in \mathbb{Z}, \\ \kappa_n = -\sum_{i=1}^{n-1} \kappa_i \end{array} \right\}.$$

PROPOSITION 5.4.3. *For $F$ and $P$ as above, the following two properties hold:*

1. *$F \cdot p \cap F \cdot q = \emptyset$ for every $p \neq q$ in $P$.*

*2.* $\bigcup_{p\in P} F\cdot p = \mathrm{GL}_n(\mathbb{R})$.

*Proof.* To simplify the notation, let $w_n = \prod_{i=1}^{n-1} w_i^{-1}$. To establish the claim, we show that for each $a \in \mathrm{GL}_n(\mathbb{R})$, the equation

$$a = sk\begin{pmatrix} w_1 & \cdots & w_1 y_{1,n-1} & w_1 y_{1,n} \\ & \ddots & & \vdots \\ & & w_{n-1} & w_{n-1}y_{n-1,n} \\ & & & w_n \end{pmatrix} 2^{\lambda} \begin{pmatrix} 2^{\kappa_1} & \cdots & 2^{\kappa_{n-1}}\mu_{1,n-1} & 2^{\kappa_n}\mu_{1,n} \\ & \ddots & & \vdots \\ & & 2^{\kappa_{n-1}} & 2^{\kappa_n}\mu_{n-1,n} \\ & & & 2^{\kappa_n} \end{pmatrix} \tag{5.10}$$

has a unique solution subject to the constraints $\lambda, \kappa_i, \mu_{i,j} \in \mathbb{Z}$, $s, w_i \in [1,2)$, $y_{i,j} \in [0,1)$.

First note that by uniqueness of Iwasawa decomposition, the element $k \in \mathrm{O}_n$ in the above equation is unique. Moreover, from (5.10) we have $|\det(a)| = (s2^{\lambda})^n$. It is not difficult to show that this implies

$$\lambda = \left\lfloor \frac{1}{n}\sum_{i=1}^{n} \log_2(a'_{i,i}) \right\rfloor, \qquad \text{and} \qquad s = 2^{-\lambda}\left(\prod_{i=1}^{n} a'_{i,i}\right)^{1/n}.$$

To proceed, let $a' = \frac{1}{|\det(a)|^{1/n}} k^{-1}a$ (noting that $k$ is uniquely determined at this point.) Clearly, $a'$ is upper triangular with determinant 1, and we are left to show that

$$a' = \begin{pmatrix} w_1 & \cdots & w_1 y_{1,n-1} & w_1 y_{1,n} \\ & \ddots & & \vdots \\ & & w_{n-1} & w_{n-1}y_{n-1,n} \\ & & & w_n \end{pmatrix} \begin{pmatrix} 2^{\kappa_1} & \cdots & 2^{\kappa_{n-1}}\mu_{1,n-1} & 2^{\kappa_n}\mu_{1,n} \\ & \ddots & & \vdots \\ & & 2^{\kappa_{n-1}} & 2^{\kappa_n}\mu_{n-1,n} \\ & & & 2^{\kappa_n} \end{pmatrix} \tag{5.11}$$

has a unique solution given the constraints $w_i \in [1,2)$, $y_{i,j} \in [0,1)$, and $\kappa_i, \mu_{i,j} \in \mathbb{Z}$. To do so, we denote the product of the two matrices on the right hand side of (5.11) as

$z = [z_{i,j}]$, and we observe that these entries have the form

$$z_{i,j} = \begin{cases} 0, & \text{for } j - i < 0, \\ w_i 2^{\kappa_i}, & \text{for } j - i = 0, \\ w_i 2^{\kappa_j}(y_{i,j} + \mu_{i,j}), & \text{for } j - i = 1, \\ w_i 2^{\kappa_j}(y_{i,j} + \mu_{i,j} + \sum_{k=i+1}^{j-1} y_{i,k}\mu_{k,j}) & \text{for } j - i > 1. \end{cases}$$

We now proceed by an induction-like argument on the diagonals, starting from the main diagonal. Namely, note that $a'_{i,i} = 2^{\kappa_i} w_i$, with constraints $w_i \in [1, 2)$ and $\kappa_i \in \mathbb{Z}$, has a unique solution for each $1 \leq i \leq n$. Moving to the super-diagonal, we likewise solve these equations to find that when $j - i = 1$, we have

$$\mu_{i,j} = \left\lfloor \frac{2^{-\kappa_j} a'_{i,j}}{w_i} \right\rfloor, \qquad \text{and} \qquad y_{i,j} = \frac{2^{-\kappa_j} a'_{i,j}}{w_i} - \left\lfloor \frac{2^{-\kappa_j} a'_{i,j}}{w_i} \right\rfloor.$$

Finally, the remaining entries when $j - i > 1$ can be computed similarly, as follows:

$$\mu_{i,j} = \left\lfloor \frac{2^{-\kappa_j} a'_{i,j}}{w_i} - \sum_{k=i+1}^{j-1} y_{i,k}\mu_{k,j} \right\rfloor, \qquad \text{and}$$

$$y_{i,j} = \frac{2^{-\kappa_j} a'_{i,j}}{w_i} - \sum_{k=i+1}^{j-1} y_{i,k}\mu_{k,j} - \left\lfloor \frac{2^{-\kappa_j} a'_{i,j}}{w_i} - \sum_{k=i+1}^{j-1} y_{i,k}\mu_{k,j} \right\rfloor,$$

where all values on the right hand sides are known from previous diagonals. This establishes the proposition. □

Note that Proposition 5.4.3 shows that the pair $(P, \overline{F})$ forms a frame generator in the sense of [11], and the pair $(F, P)$ satisfies the slightly different definition of a tiling system given in [58]. We now show that $(F, P)$ also fulfills the new conditions for a tiling system given in Definition 5.3.1.

Corollary 5.4.4. *For $F$ and $P$ as given in the previous proposition, we have*

(i) $\lambda_{\mathrm{GL}_n}(\overline{F} \cdot p \cap \overline{F} \cdot q) = 0$ *for every distinct pair* $p, q \in P$,

(ii) $\lambda_{\mathrm{GL}_n}(\mathrm{GL}_n(\mathbb{R}) \setminus \bigcup\{\overline{F} \cdot p \ : \ p \in P\}) = 0.$

*Proof.* Property (ii) is immediate, as

$$\mathrm{GL}_n(\mathbb{R}) = \bigcup \{\, F \cdot p \;\mid\; p \in P \,\} \subseteq \bigcup \{\, \overline{F} \cdot p \;\mid\; p \in P \,\} \subseteq \mathrm{GL}_n(\mathbb{R}).$$

For the first property, note that by Proposition 5.4.3, $F \cdot p \cap F \cdot q = \emptyset$ if $p \neq q \in P$. So,

$$\begin{aligned}
\lambda_{\mathrm{GL}_n(\mathbb{R})}(\overline{F} \cdot p \cap \overline{F} \cdot q) &= \lambda_{\mathrm{GL}_n(\mathbb{R})}\big((\overline{F} \setminus F) \cdot p \cap \overline{F} \cdot q\big) \cup \big(\overline{F} \setminus F) \cdot q \cap \overline{F} \cdot p)\big) \\
&\leq \lambda_{\mathrm{GL}_n(\mathbb{R})}\big((\overline{F} \setminus F) \cdot p\big) + \lambda_{\mathrm{GL}_n(\mathbb{R})}\big((\overline{F} \setminus F) \cdot q\big) \\
&= \lambda_{\mathrm{GL}_n(\mathbb{R})}(\overline{F} \setminus F) + \lambda_{\mathrm{GL}_n(\mathbb{R})}(\overline{F} \setminus F) \\
&= 0,
\end{aligned}$$

where the second to last equality follows as $\mathrm{GL}_n(\mathbb{R})$ is unimodular, and the last equality can be verified computed directly using Theorem 5.4.1. □

We now show how to construct an open set $F_{\mathrm{o}}$ containing $\overline{F}$. Fix $\epsilon > 0$. Let $F_{\mathrm{o}}$ be the set

$$F_{\mathrm{o}} = \left\{ sk \begin{pmatrix} w_1 & w_1 y_{1,2} & w_1 y_{1,3} & \cdots & w_1 y_{1,n-1} & w_1 y_{1,n} \\ & w_2 & w_2 y_{2,3} & \cdots & w_2 y_{2,n-1} & w_2 y_{2,n} \\ & & w_3 & \cdots & w_3 y_{3,n-1} & w_3 y_{3,n} \\ & & & \ddots & & \vdots \\ & & & & w_{n-1} & w_{n-1} y_{n-1,n} \\ & & & & & \prod_{i=1}^{n-1} w_i^{-1} \end{pmatrix} \;\middle|\; \begin{array}{r} s, w_i \in (1-\epsilon, 2+\epsilon), \\ y_{i,j} \in (-\epsilon, 1+\epsilon), \\ k \in \mathrm{O}_n \end{array} \right\}.$$

PROPOSITION 5.4.5. *$F_{\mathrm{o}}$ is an open set in $\mathrm{GL}_n(\mathbb{R})$ such that $\overline{F} \subseteq F_{\mathrm{o}}$.*

*Proof.* The inclusion $\overline{F} \subseteq F_{\mathrm{o}}$ is clear. To prove that $F_{\mathrm{o}}$ is open in $\mathrm{GL}_n(\mathbb{R})$, note that the multiplication map

$$\mathrm{SO}_n \times \mathrm{D}_n \times \mathrm{T}_n \to \mathrm{SL}_n(\mathbb{R})$$

is a diffeomorphism (see Proposition 1.6.2 of [1]). Consequently, the multiplication map $\mathbb{R}^+ \times \mathrm{O}_n \times \mathrm{D}_n \times \mathrm{T}_n \to \mathrm{GL}_n(\mathbb{R})$ is a homeomorphism and maps any open set to an open subset of $\mathrm{GL}_n(\mathbb{R})$. □

Summarizing the results from this section, we have constructed a tiling system $(F, P)$ and determined how to compute the measure of a set $E$ through the coordinates of the Iwasawa decomposition, which we used to motivate the tiling. We now use these results to compute frame bounds for the discretization of the continuous wavelet transform.

## 5.5 Computation of Frame Bounds

In this section, we find bounds for the value $M$ (as defined for Theorem 5.3.6) associated with the sets $P, F$, and $F_\mathrm{o}$. Recall that $M$ is just the least uniform upper bound for the number of $q \in P$ such that for a fixed $p \in P$, $\overline{F} \cdot p$ and $F_\mathrm{o} \cdot q$ have non-trivial intersection. Once we have done this for general $n$, we provide a concrete example with all details when $n = 2$.

PROPOSITION 5.5.1. *For $F$, $P$, and $F_\mathrm{o}$ defined in Section 5.4, $M$ as in Notation 5.3.5, and $0 < \epsilon \leq \frac{1}{2}$, we have*

$$M = \sup_{p \in P} \left| \left\{ k \in P \;\middle|\; \overline{F} \cdot p \cap F_\mathrm{o} \cdot k \neq \emptyset \right\} \right| \leq 3^n 6^{\frac{n(n-1)}{2}}.$$

*Proof.* If $\overline{F} \cdot p \cap F_\mathrm{o} \cdot p' \neq \emptyset$ for some $p \neq p' \in P$, then there exist $a \in \overline{F}$ and $a' \in F_\mathrm{o}$ for which we have $ap = a'p'$, or equivalently

$$a = a'p'p^{-1}. \tag{5.12}$$

Using Theorem 5.4.1, we write the Iwasawa decompositions for both $a = skwy$ and $a' = s'k'w'y'$, where $w$ and $w'$ are diagonal matrices $\mathrm{diag}(w_1, \ldots, w_{n-1}, (w_1 \cdots w_{n-1})^{-1})$ and $\mathrm{diag}(w'_1, \ldots, w'_{n-1}, (w'_1 \cdots w'_{n-1})^{-1})$, and $y = [y_{i,j}]$ and $y' = [y'_{i,j}]$ are unit upper triangular matrices. Suppose $p, p'$ are written in the format of $P$ using $\lambda, \kappa_i, \mu_{i,j}$ and $\lambda', \kappa'_i, \mu'_{i,j}$ respectively. By the uniqueness of Iwasawa decomposition and equality of the determinants of both sides of the above equation, we get:

$$k = k', \tag{C1}$$

$$s = s' 2^{\lambda' - \lambda}. \tag{C2}$$

From (5.12), we get

$$wy = w'y'\,[\mu'_{i,j}]\,\mathrm{diag}(2^{\kappa'_1-\kappa_1},\ldots,2^{\kappa'_n-\kappa_n})\,[\mu_{i,j}]^{-1}, \tag{5.13}$$

which simplifies to

$$wy = \underbrace{w'\,\mathrm{diag}(2^{\kappa'_1-\kappa_1},\ldots,2^{\kappa'_n-\kappa_n})}_{\in \mathrm{D}_n}\,\underbrace{\mathrm{diag}(2^{-\kappa'_1+\kappa_1},\ldots,2^{-\kappa'_n+\kappa_n})\,y'\,[\mu'_{i,j}]\,\mathrm{diag}(2^{\kappa'_1-\kappa_1},\ldots,2^{\kappa'_n-\kappa_n})\,[\mu_{i,j}]^{-1}}_{\in \mathrm{T}_n},$$

where $y, y', [\mu_{i,j}], [\mu'_{i,j}] \in \mathrm{T}_n$ are unit upper triangular matrices and $\sum_{i=1}^{n}\kappa_i = \sum_{i=1}^{n}\kappa'_i = 0$ (as in the definition of $P$). Note that in the above equation, we used the fact that $\mathrm{D}_n$ normalizes $\mathrm{T}_n$. By uniqueness of Iwasawa decomposition, we have

$$w = w'\,\mathrm{diag}(2^{\kappa'_1-\kappa_1},\ldots,2^{\kappa'_n-\kappa_n}),$$

and

$$[y_{i,j}] = \mathrm{diag}(2^{-\kappa'_1+\kappa_1},\ldots,2^{-\kappa'_n+\kappa_n})\,[y'_{i,j}]\,[\mu'_{i,j}]\,\mathrm{diag}(2^{\kappa'_1-\kappa_1},\ldots,2^{\kappa'_n-\kappa_n})\,[\mu_{i,j}]^{-1}.$$

Thus we obtain the conditions

$$w'_i 2^{\kappa'_i-\kappa_i} = w_i \qquad \forall i \tag{C3}$$

$$2^{(\kappa'_j-\kappa_j)-(\kappa'_i-\kappa_i)}\sum_{k=i}^{j} y'_{i,k}\mu'_{k,j} = \sum_{k=i}^{j} y_{i,k}\mu_{k,j} \qquad \text{for } j \geq i \tag{C4}$$

by comparing each product entry-wise.

Given a fixed $p \in P$ represented by $\lambda, \kappa_i, \mu_{i,j}$, we count the number of $p' \in P$ with parameters $\lambda', \kappa'_i, \mu'_{i,j}$ for which (5.13) can be satisfied for some choice of $a \in \overline{F}$ and $a' \in F_{\mathrm{o}}$. Using condition (C1), we see that

$$s' 2^{\lambda'-\lambda} = s \in [1,2].$$

As $s' \in (1-\epsilon, 2+\epsilon)$, if $\epsilon \leq \frac{1}{2}$, we deduce that $\lambda'-\lambda \in \{-1,0,1\}$. This also shows that $\lambda'-\lambda = -1$ implies $s' \in [2, 2+\epsilon)$, $\lambda'-\lambda = 0$ implies $s' \in [1,2]$, and finally that $\lambda'-\lambda = 1$ implies that $s' \in (1-\epsilon, 1]$. Similarly, we deduce from (C3) that

$\kappa_i' - \kappa_i \in \{-1, 0, 1\}$ holds since $\epsilon \le \frac{1}{2}$. This constrains $w_i'$ to be in $[2, 2+\epsilon)$, $[1, 2]$ or $(1-\epsilon, 1]$ respectively. Also, note that for a given choice of $\kappa_i' \in \{\kappa_i - 1, \kappa_i, \kappa_i + 1\}$, we have a uniquely determined $w_i'$ given by $w_i' = \frac{w_i}{2^{\kappa_i' - \kappa_i}}$.

From this point forward, we assume that $\lambda'$ and $\kappa_1', \ldots, \kappa_{n-1}'$ have been chosen according to the above constraints, and the parameters $s'$, $w_i'$ and $\kappa_n'$ have been determined accordingly; recall that $\kappa_n' = -\sum_{i=1}^{n-1} \kappa_i'$. Let $p_{i,j} = (\kappa_j' - \kappa_j) - (\kappa_i' - \kappa_i)$. As $\kappa_j' - \kappa_j$, $\kappa_i' - \kappa_i \in \{-1, 0, 1\}$, it follows that $p_{i,j} \in \{-2, -1, 0, 1, 2\}$. To count possible solutions for equations, we will make repeated use of the following claim.

Claim 5.5.2. *Let $0 < \epsilon \le \frac{1}{2}$ be fixed. Then for any interval $[\alpha, \alpha+4] \subset \mathbb{R}$, there are at most six $\beta \in \mathbb{Z}$ such that $[\alpha, \alpha+4] \cap (\beta - \epsilon, \beta + 1 + \epsilon) \neq \emptyset$.*

*Proof of Claim.* Let $m = \min\{\, \beta \in \mathbb{Z} \mid [\alpha, \alpha+4] \cap (\beta - \epsilon, \beta + 1 + \epsilon) \neq \emptyset \,\}$. Clearly $m$ exists and is finite, as both intervals are bounded and $\mathbb{Z}$ is discrete. Now, consider $m + k$ for $k \ge 6$. If $(m + k - \epsilon, m + k + 1 + \epsilon)$ intersects $[\alpha, \alpha + 4]$, then we have $(m - \epsilon, m + 1 + \epsilon)$ intersects $[\alpha - k, \alpha - k + 4]$. On the other hand, by definition of $m$, we know that $(m - \epsilon, m + 1 + \epsilon)$ intersects $[\alpha, \alpha + 4]$ as well. This is a contradiction, because $(m - \epsilon, m + 1 + \epsilon)$ has length at most 2, which is not larger than the gap between the above two closed intervals. Therefore, there are at most 6 possible choices for $\beta \in \mathbb{Z}$ for which we may have $[\alpha, \alpha + 4] \cap (\beta - \epsilon, \beta + 1 + \epsilon) \neq \emptyset$. □

Returning to our main proof of Proposition 5.5.1, we now proceed by an inductive argument. We examine diagonals of the matrices on the two sides of (C4), starting from the super-diagonal. For $j = i + 1$, condition (C4) becomes

$$y_{i,i+1}' + \mu_{i,i+1}' = 2^{-p_{i,i+1}}(\mu_{i,i+1} + y_{i,i+1}),$$

with the constraints $y_{i,i+1}' \in (-\epsilon, 1 + \epsilon)$, $y_{i,i+1} \in [0, 1]$ and $\mu_{i,i+1}' \in \mathbb{Z}$. Note that $2^{-p_{i,i+1}}\mu_{i,i+1}$ is fixed at this point, and clearly $2^{-p_{i,i+1}}y_{i,i+1} \in [0, 4]$. So by Claim 5.5.2, there are at most six possible choices for $\mu_{i,i+1}'$.

Next, by induction hypothesis, assume that for every $\mu_{i,j}'$ with $j - i < k$, there are at most six possible choices that satisfy (C4). Also assume that $y_{i,j}, y_{i,j}'$ and $\mu_{i,j}'$

have been fixed whenever $j-i<k$ (*i.e.*, for the first $k-1$ diagonals). For $j=i+k$, condition (C4) becomes

$$y'_{i,i+k}+\mu'_{i,i+k}=2^{-p_{i,i+k}}y_{i,i+k}+\underbrace{\left(2^{-p_{i,i+k}}\sum_{t=i}^{k+i-1}y_{i,t}\mu_{t,i+k}-\sum_{t=i+1}^{k+i-1}y'_{i,t}\mu'_{t,i+k}\right)}_{\alpha}, \tag{5.14}$$

where $\mu'_{i,i+k}\in\mathbb{Z}$, $y'_{i,i+k}\in(-\epsilon,1+\epsilon)$ and $y_{i,i+k}\in[0,1]$. Note that the expression $\alpha$ in the above equation is fixed, as it only involves values which have already been chosen. So, by Claim 5.5.2, there are at most six possible choices for $\mu'_{i,i+k}$. This proves that for every $\mu'_{i,j}$ in (C4), there are only six possible values that may satisfy the equation.

In the following table, we summarize what we have found so far. Note that not every possible solution is necessarily an actual solution; however, any solution for (C4) is counted below.

| parameter | number of possible choices | possible solutions in terms of $p$ |
|---|---|---|
| $\lambda'$ | 3 | $\lambda-1,\lambda,\lambda+1$ |
| $\kappa'_i, 1\leq i<n$ | 3 | $\kappa_i-1,\kappa_i,\kappa_i+1$ |
| $\kappa'_n$ | 1 | $-\sum_{i=1}^{n-1}\kappa'_i$ |
| $\mu'_{i,j}, i<j$ | 6 | solutions to (5.14) satisfying given constraints |

**Table 5.1:** Maximum number of possible choices for each parameter in $p'$.

Combining these results, we conclude that for a fixed $p\in P$, there are at most $3^n6^{\frac{n(n-1)}{2}}$ possible choices for $p'\in P$ such that $\overline{F}\cdot p\cap F_\circ\cdot p'\neq\emptyset$. □

Remark 5.5.3. The case for $n=2$ was also previously studied in [58]. Proposition 5.5.1, together with Theorem 5.3.6, shows that using our methods one can construct frames with significantly better frame condition numbers than the construction in [58]. Indeed, the ratio of the frame bounds in our construction is $\frac{C_2}{C_1}\leq 54$, whereas the frame condition number $\frac{C_2}{C_1}$ for the construction in [58] was about 1782. (In fact, it is only mentioned in [58] that $M<\infty$. However, looking into their arguments closely, one can obtain that $M$ is bounded by 1782. Also, note that there is a typo in the definition of

$M$ in [58]; the correct formula should be $M = \sup_{p \in P} \left| \left\{ p' \in P \;\middle|\; p \cdot D \cap p' \cdot D \neq \emptyset \right\} \right|$.) As the rate of convergence in frame calculations when approximating signals is highly sensitive to the frame condition number (*i.e.*, the ratio of the frame bounds), our methods result in much more practical and efficient frames than those of [58] for the case of $n = 2$.

Remark 5.5.4. When $n = 2$, the above proposition gives $M \leq 54$. However, in this case we can obtain the following improvement.

Corollary 5.5.5. *In the same setting as Proposition 5.5.1, if $n = 2$, then $M \leq 36$.*

*Proof.* When $n = 2$, we have

$$p' = 2^{\lambda'} \begin{bmatrix} 2^{\kappa'} & 2^{-\kappa'}\mu' \\ 0 & 2^{-\kappa'} \end{bmatrix}.$$

Therefore there is only one one $\kappa'$ and $\mu'$ to choose, and $p_{1,2} \in \left\{-2, 0, 2\right\}$. By bounding $\mu'$ for each $p_{1,2}$ separately, we can obtain the improved bound.

By Claim 5.5.2, we still have at most 6 choices when $p_{1,2} = -2$. Adapting Claim 5.5.2 in the case $p_{1,2} = 0$, (intervals $[\alpha, \alpha + 1]$), there are at most 3 choices for $\mu'$. Finally, for the case $p_{1,2} = 2$ (intervals $[\alpha, \alpha + \frac{1}{4}]$), we have for $0 < \epsilon < \frac{1}{4}$ there are at most 2 choices of $\mu'$, and for $\frac{1}{4} \leq \epsilon < \frac{1}{2}$ there are at most 3 choices for $\mu'$. So based on $0 < \epsilon < \frac{1}{4}$ or $\frac{1}{4} \leq \epsilon < \frac{1}{2}$, one gets $M \leq 33$ and $M \leq 36$, respectively. □

Similar improvements could be made for higher-dimensional cases, but finding the improved bounds in these cases becomes quite complicated and currently provides little additional value.

## 5.6 Concrete Frame Construction for $n = 2$

So far, we have established that the proposed sets $F, F_o$, and $P$ satisfy the conditions of Theorem 5.3.6, and have calculated $M$ as well. In this section, we use these sets and follow the construction in Theorem 5.3.6, to give an explicit example of a discrete frame for $L^2(\mathrm{M}_2(\mathbb{R}))$. A similar approach can be taken for higher dimensions, but we focus our attention on $n = 2$ for now.

The Iwasawa decomposition for $\mathrm{GL}_2(\mathbb{R})$ can be stated as follows: Let $\mathrm{O}_2$ denote the group of orthogonal $2 \times 2$ matrices, $\mathrm{D}_2$ denote the diagonal $2 \times 2$ matrices with positive diagonal entries and determinant 1, and $\mathrm{T}_2$ denote the $2 \times 2$ unit upper triangular matrices. Every element of $\mathrm{GL}_2(\mathbb{R})$ can be uniquely decomposed as an ordered product of elements in $\mathrm{O}_2$, $\mathrm{D}_2$, and $\mathrm{T}_2$. That is, $\mathrm{GL}_2(\mathbb{R}) = \mathrm{O}_2\mathrm{D}_2\mathrm{T}_2$. Note that $\mathrm{O}_2$ is compact, and $\mathrm{T}_2$ and $\mathrm{D}_2$ are both abelian subgroups of $\mathrm{GL}_2(\mathbb{R})$.

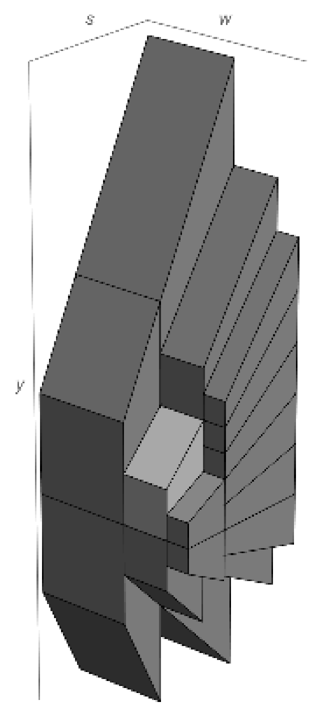



**Figure 5.1:** Multiscale tiling based on $(F, P)$. The base tile is lighter. Several shifted tiles are shown in darker gray.

**Tiling System.** Let $P = \left\{ 2^\lambda \begin{pmatrix} 2^\kappa & 2^{-\kappa}\mu \\ 0 & 2^{-\kappa} \end{pmatrix} \;\middle|\; \lambda, \kappa, \mu \in \mathbb{Z} \right\}$,

$$F = \left\{ \begin{pmatrix} \pm\cos\theta & \mp\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} sw & swy \\ 0 & sw^{-1} \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi),\ s, w \in [1, 2),\ y \in [0, 1) \right\}.$$

Then $(F, P)$ forms a tiling system in the sense of Definition 5.3.1 for $\mathrm{GL}_2(\mathbb{R})$. After projecting on the $(s, w, y)$-space, this tile and some of its translates under the action of $P$ are shown in Figure 5.1. Note that by Corollary 5.5.5, we have $M \leq 36$.

As before, we take $0 < \epsilon \leq \frac{1}{2}$, and define $F_\mathrm{o}$ to be

$$F_\mathrm{o} = \left\{ \begin{pmatrix} \pm\cos\theta & \mp\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} sw & swy \\ 0 & sw^{-1} \end{pmatrix} \;\middle|\; \begin{array}{r} \theta \in [0, 2\pi) \\ s, w \in (1-\epsilon, 2+\epsilon) \\ y \in (-\epsilon, 1+\epsilon) \end{array} \right\}.$$

**Reconstruction Formula.** Recall the explicit formulation of the reconstruction formula (5.8) for $n = 2$. The wavelet condition (Equation (5.15)) and reconstruction formula (Equation (5.16)) were obtained in Theorem 2.1 of [58].

Let $\psi \in L^2(\mathbb{R}^4)$. If

$$\int_{\mathbb{R}^4} \left| \widehat{\psi}(h_1, h_2, h_3, h_4) \right|^2 \frac{dh_1 dh_2 dh_3 dh_4}{|h_1 h_4 - h_2 h_3|^2} = 1, \tag{5.15}$$

then $\psi$ is a wavelet. For $x, y \in \mathrm{M}_2(\mathbb{R})$ and $h \in \mathrm{GL}_2(\mathbb{R})$, define $\psi^{x,h}(y)$ as

$$\frac{1}{a}\psi\left( \frac{h_4(y_1-x_1)-h_2(y_3-x_3), h_4(y_2-x_2)-h_2(y_4-x_4), h_1(y_3-x_3)-h_3(y_1-x_1), h_1(y_4-x_4)-h_3(y_2-x_2)}{h_1 h_4 - h_2 h_3} \right),$$

where $a = |h_1 h_4 - h_2 h_3|$. Then, for any $f \in L^2(\mathbb{R}^4)$, we have

$$f = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \langle\, f, \psi^{x,h} \,\rangle\, \psi^{x,h} \; \frac{dx_1 \cdots dx_4\, dh_1 \cdots dh_4}{|h_1 h_4 - h_2 h_3|^4}, \tag{5.16}$$

weakly in $L^2(\mathbb{R}^4)$. Conversely, if (5.16) holds for every $f \in L^2(\mathbb{R}^4)$, then $\psi$ is a wavelet.

We now conclude this chapter by describing the discretization of (5.16) to make it computationally feasible.

**The Discrete Frame.** To discretize the continuous frame, we find a 4-dimensional cube $R$ containing $F_{\mathrm{o}}$. Suppose

$$R = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; x_i \in [a_i, b_i] \;\text{ for }\; i \in \{1,2,3,4\} \right\},$$

where $a_i < b_i$, for $1 \le i \le 4$, are fixed real numbers. We need to determine appropriate values for $a_i$ and $b_i$ so that $F_{\mathrm{o}} \subseteq R$. Consider an arbitrary element of $F_{\mathrm{o}}$ together with its Iwasawa decomposition, say

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \pm\cos\theta & \mp\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} sw & swy \\ 0 & \frac{s}{w} \end{pmatrix},$$

where $s, w \in (1-\epsilon, 2+\epsilon)$ and $y \in (-\epsilon, 1+\epsilon)$. Comparing the two sides of the above matrix equation, we get

$$\begin{aligned} |x_1|, |x_3| &\le |sw| \le (2+\epsilon)^2 < 7, \\ |x_2|, |x_4| &< \sqrt{|swy|^2 + |\frac{s}{w}|^2} \le (2+\epsilon)(1+\epsilon)\sqrt{w^2 + \frac{1}{w^2}} \le \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\sqrt{\frac{13}{2}} < 10, \end{aligned}$$

where we used the fact that $\epsilon \le \frac{1}{2}$. Thus, we can set $a_1 = a_3 = -7$, $b_1 = b_3 = 7$, $a_2 = a_4 = -10$, and $b_2 = b_4 = 10$.

Let $L^2(R)$ be the closed subspace of $L^2(\mathbb{R}^4)$ consisting of all the elements supported on $R$. We can construct an orthonormal basis of $L^2(R)$ indexed by the set $J$ defined as

$$J = \left\{ \lambda = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_3 \in \frac{1}{14}\mathbb{Z},\; \lambda_2, \lambda_4 \in \frac{1}{20}\mathbb{Z} \right\}.$$

Then for all $g \in L^2(\mathrm{M}_2(\mathbb{R}))$ which satisfy $\mathbb{1}_{\overline{F}} \le \widehat{g} \le \mathbb{1}_{F_{\mathrm{o}}}$, we have

$$\left\{ \rho[\lambda, p]^{-1} g \;\middle|\; (\lambda, p) \in J \times P \right\}$$

is a discrete frame with frame bounds $C_1 = |R|, C_2 = |R|M$. That is

$$|R|\,\|f\|^2 \le \sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, f, \rho[\gamma,k]^{-1}g\,\rangle_{L^2(\mathrm{M}_2(\mathbb{R}))}\right|^2 \le |R|\,M\,\|f\|^2$$

for all $f \in L^2(\mathrm{M}_2(\mathbb{R}))$.

REMARK 5.6.1. Note that $|R|$ in the above construction is $14^2 \times 20^2$, which is far smaller than the similar parameter from [58], which was $176^4$. Also, when it is necessary or useful, one can easily compute $\rho[\lambda,p]^{-1}g$ explicitly to obtain a formula similar to $\psi^{x,h} = \rho[x,h]\psi$. In fact, in Section 5.8, where we discuss applications to more general function spaces, this will appear in the proof of Theorem 5.8.1.

## 5.7 Improved Bounds for Subclasses of Signals

For some classes of signals, we can improve the frame bounds by introducing a dependence on the measures of the overlapping tiles. Towards that end, we first prove an upper bound on the measure of intersecting sets for our tiling system introduced in Section 5.4. We prove the general bound for the case $n \ge 3$, and will compute explicitly the upper bound for $n = 2$ in the example below, which proves to be a special case. We will then conclude by finding conditions on signals for which improved bounds hold for the discrete frame in 5.3.6.

THEOREM 5.7.1. *Let $F$, $P$, and $F_\mathrm{o}$ be as defined in Section 5.4 and let $E_{p,p'}$ denote the set $\overline{F}\cdot p \cap F_\mathrm{o}\cdot p'$. Then for $n \ge 3$ and $\epsilon \le (2^{2n-3}-1)^{\frac{1}{2n-4}} - 2$, the bound*

$$\lambda_{\mathrm{GL}_n}(E_{p,p'}) \le c_n \left[\frac{(2+\epsilon)^{2n-2} - 2^{2n-2}}{2n-2}\right]$$

*holds, where the constant is given by*

$$c_n = \log(2)\prod_{i=2}^{n-2} \frac{2^{2n-2i}-1}{2n-2i}.$$

*Proof.* Here we provide an overview of the argument. We then prove each statement rigorously below.

**Claim 1:** First, we establish an upper bound can be obtained by integrating each variable separately and multiplying the results.

**Step 2:** Now recall by Proposition 5.5.1, we found that for any given $E_{p,p'}$, we must have $s$ and $w_i$ ($1 \leq i < n$) each limited to one of the intervals $(1-\epsilon, 1]$, $[1, 2]$, or $[2, 2+\epsilon)$. Similarly we had $y_{i,j}$ in one of the intervals $(-\epsilon, 0]$, $[0, 1]$, or $[1, 1+\epsilon)$ for each $i < j \leq n$. So we define the functions

$$f_n^{(1)}(i) = \frac{\int_{1-\epsilon}^{1} w^{2n-2i-1}\, dw}{\int_1^2 w^{2n-2i-1}\, dw}, \qquad f_n^{(2)}(i) = \frac{\int_{2}^{2+\epsilon} w^{2n-2i-1}\, dw}{\int_1^2 w^{2n-2i-1}\, dw},$$

$$g^{(1)} = \frac{\int_{1-\epsilon}^{1} s^{-1}\, ds}{\int_1^2 s^{-1}\, ds}, \qquad g^{(2)} = \frac{\int_{2}^{2+\epsilon} s^{-1}\, ds}{\int_1^2 s^{-1}\, ds},$$

$$h_{i,j}^{(1)} = \frac{\int_{-\epsilon}^{1} dy_{i,j}}{\int_0^1 dy_{i,j}}, \qquad h_{i,j}^{(2)} = \frac{\int_{1}^{1+\epsilon} dy_{i,j}}{\int_0^1 dy_{i,j}}.$$

Note that each function produces the ratio between the integral when corresponding parameters differ, but possibly overlap, as summarized in Table 5.1.

**Claim 3:** For a fixed dimension $n$ and $\epsilon(n)$ sufficiently small, each ratio above is less than or equal to 1. This shows that the integral over $[0, 1]$ or $[1, 2]$ results in the larger value for each possible value, so the maximum product would be over those in each case—however, this only occurs if $p = p'$.

**Claim 4:** As $p \neq p'$, we must determine which single parameter change results in the largest product, but this is the same as maximizing over all the ratio functions above for each fixed dimension $n$. So finally we prove that the maximum over all other parameters occurs for $f_n^{(2)}(1)$.

**Conclusion:** This establishes the final result, as it shows that the maximum value of the product occurs for $p \neq p'$ when all parameters are equal except for $\kappa_1' = \kappa_1 - 1$, and the bound is

$$\log(2) \prod_{i=2}^{n-2} \frac{2^{2n-2i}-1}{2n-2i} \left[\frac{(2+\epsilon)^{2n-2} - 2^{2n-2}}{2n-2}\right].$$

*Proof of Claim 1.* Let $p, p' \in P$ be such that $E = \left(\overline{F} \cdot p \cap F_{\mathrm{o}} \cdot p'\right) \neq \emptyset$ and $p \neq p'$. Then to compute an upper bound on the measure of $E_{p,p'}$, first note that for $a \in \overline{F}$ and $a' \in F_{\mathrm{o}}$, we have

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{p,p'}) = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \mathbb{1}_E(a)\, da = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \mathbb{1}_E(a'p'p^{-1})\, d(a'p'p^{-1}) = \int\limits_{\mathrm{GL}_n(\mathbb{R})} \mathbb{1}_E(a'p'p^{-1})\, da'.$$

We next use the change of variables introduced in Theorem 5.4.1, $a' \mapsto kswy$, and note that $\overline{F}$ and $F_{\mathrm{o}}$ are both invariant under actions of $\mathrm{O}_n$ on the left (by definition), so that $\mathbb{1}_E(kswyp'p^{-1}) = \mathbb{1}_E(swyp'p^{-1})$. This demonstrates that

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{p,p'}) = \int\limits_{\mathrm{O}_n \times \mathbb{R}^* \times \mathrm{D_n} \times \mathrm{T_n}} \mathbb{1}_{E_{p,p'}}(swyp'p^{-1}) s^{-1} \prod_{i=1}^{n-1} w_i^{2(n-i)-1}\, dk\, ds \prod_{i=1}^{n-1} dw_i \prod_{i \leq j} dy_{i,j},$$

where $dk$ is the normalized Haar measure on $\mathrm{O}_n$. Additionally, $ds$, $dw_i$, $dy_{i,j}$ are each Lebesgue measure on $\mathbb{R}$ for all $i, j$.

Finally, noting that $\mathbb{1}_{E_{p,p'}}(swyp'p^{-1}) \leq \mathbb{1}_{I^{(1)}}(s) \mathbb{1}_{I_i^{(2)}}(w_i) \mathbb{1}_{I_{i,j}^{(3)}}(y_{i,j})$, where each interval $I^{(\cdot)}$ is determined by the choice of $p$ and $p'$ following the work done in Proposition 5.5.1, wherein we obtained conditions on the only possible parameters in the $swy$ decomposition for which $\mathbb{1}_E(swyp'p^{-1})$ can be non-zero. This demonstrates

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{p,p'}) \leq \left(\int_{I^{(1)}} s^{-1}\, ds\right) \left(\prod_{i=1}^{n-1} \int_{I_i^{(2)}} w_i^{2(n-i)-1}\, dw_i\right) \left(\prod_{i \leq j} \int_{I_{i,j}^{(3)}} dy_{i,j}\right),$$

where the possible intervals for $I^{(1)}$, $I_i^{(2)}$, and $I_{i,j}^{(3)}$ were established in Proposition 5.5.1, and used in the definitions of the functions in Step 2. □

*Proof of Claim 3.* It is easy to compute the cases associated with $y_{i,j}$ and $s$ as

$$g^{(1)} = \log_2\left(\frac{1}{1-\epsilon}\right), \qquad g^{(2)} = \log_2\left(\frac{2+\epsilon}{2}\right).$$

It is easy to show that for $\epsilon \in (0, \frac{1}{2}]$, we have that $g^{(1)} \geq h^{(1)}_{i,j} = h^{(2)}_{i,j} \geq g^{(2)}$ by standard techniques from calculus, which is identical to proving $f_3^{(1)}(i) \leq f_3^{(2)}(i)$ below, so we omit the details here.

Next, note that

$$f_n^{(k)}(i) = \frac{\int_{I^{(k)}} w^{2n-2i-1} dw}{\int_1^2 w^{2n-2i-1} dw} = \frac{\int_{I^{(k)}} w^{2(n-1)-2(i-1)-1} dw}{\int_1^2 w^{2(n-1)-2(i-1)-1} dw} = f_{n-1}^{(k)}(i-1), \qquad k = 1, 2, 3.$$

In particular, we have the recursive relation $f_n^{(\cdot)}(i) = f_3^{(\cdot)}(i-(n-3))$, so without loss of generality, we restrict our attention to the case $n = 3$. However, we extend the domain of definition of $f_3^{(\cdot)}$ to all integers less than or equal to 2. In particular, we show the following three facts:

1. $f_3^{(1)}(i) \leq f_3^{(2)}(i)$,
2. $g^{(1)} \leq f_3^{(2)}(1)$,
3. and $f_3^{(2)}(k+1) \leq f_3^{(2)}(k)$ for $k \in (-\infty, 1]$.

The first statement follows by defining $r(\epsilon) = (2+\epsilon)^{6-2i} - 2^{6-2i} + (1+\epsilon)^{6-2i} - 1$. Then $r(0) = 0$ and $r'(\epsilon) \geq 0$ for all $\epsilon > 0$, so the function is non-negative on $[0, \infty)$. It is straight-forward to verify that the inequality in the first statement reduces to this fact. Similarly, by defining $t(\epsilon) = f_3^{(2)}(1) - g^{(1)}$, and applying the same tests, we see that $t(\epsilon) > 0$ on $[0, \frac{1}{2}]$, which proves the second statement.

The third statement follows by induction on $k$. The base case will be $k = 1$, then

$$f_3^{(2)}(2) - f_3^{(2)}(1) = -\frac{1}{15}\epsilon(1+\epsilon)(3+\epsilon)(4+\epsilon) < 0, \qquad \forall \epsilon > 0.$$

Note that after algebraic simplification, the statement becomes

$$4^k \left[1 + \frac{\epsilon}{3}(4+\epsilon)\right] \leq 4^2 \left[(2+\epsilon)^{2k-4} + \frac{\epsilon}{3}(4+\epsilon)\right],$$

which is what we will prove in the induction step. By the induction hypothesis, we have

$$4^{k-1}\left[1+\frac{\epsilon}{3}(4+\epsilon)\right] \leq \frac{1}{4}4^2\left[(2+\epsilon)^{2k-4}+\frac{\epsilon}{3}(4+\epsilon)\right].$$

And clearly, $(2+\epsilon)^{2k-6} \leq 1$ for all $k \leq 3$ and all $0 < \epsilon \leq \frac{1}{2}$, so

$$\begin{aligned}
(2+\epsilon)^{2k-6} &\leq 1 \\
(2+\epsilon)^{2k-6}[(2+\epsilon)^2-4] &\leq \epsilon(4+\epsilon) \\
(2+\epsilon)^{2k-4}-4(2+\epsilon)^{2k-6} &\leq \epsilon(4+\epsilon) \\
(2+\epsilon)^{2k-4}+\frac{\epsilon}{3}(4+\epsilon) &\leq 4(2+\epsilon)^{2k-6}+\frac{4}{3}\epsilon(4+\epsilon) \\
4\left[(2+\epsilon)^{2k-4}+\frac{\epsilon}{3}(4+\epsilon)\right] &\leq 4^2\left[(2+\epsilon)^{2k-6}+\frac{\epsilon}{3}(4+\epsilon)\right]
\end{aligned}$$

Therefore

$$4^{k-1}\left[1+\frac{\epsilon}{3}(4+\epsilon)\right] \leq 4^2\left[(2+\epsilon)^{2k-6}+\frac{\epsilon}{3}(4+\epsilon)\right].$$

This establishes the fact that $f_n^{(2)}(1)$ is always the largest ratio for $n \geq 3$. □

As we are attempting to maximize the product of the integral terms, and we must replace one term with a smaller one, we want to choose the one with the largest ratio between terms. Combining all the previous statements, we have that maximum occurs for $f_n^{(2)}(1)$ for $n \geq 3$. This completes the proof. □

Remark 5.7.2. It is not difficult to modify the above argument to the measure on $\mathrm{M}_n(\mathbb{R})$, if such a bound proves useful. The two measures differ only by a factor of the determinant, so the only function changes will be to $g^{(1)}$ and $g^{(2)}$.

**Bound for Overlapping Tiles for $n = 2$.** This (mostly) follows from the general case presented in Theorem 5.7.1. However, it differs slightly due to the fact (in the notation presented there) $f_2^{(2)}(1) \leq g^{(1)}$. So when $n = 2$, the upper bound on sets $E$ of the form $\overline{F}\cdot p \cap F_{\circ}\cdot p'$, with $p \neq p' \in P$, is given by

$$\lambda_{\mathrm{GL}_n(\mathbb{R})}(E) \leq \frac{2^{4-2}-1}{4-2}\log\left(\frac{1}{1-\epsilon}\right) = 6\pi\log\left(\frac{1}{1-\epsilon}\right).$$

Given the reduced number of cases when $n = 2$, we have also explicitly computed all the measures for this case in Table 5.2 at the end of this chapter.

**Improved Frame Bounds.** We now show that for certain classes of signals, the bounds in 5.3.6 can be improved.

PROPOSITION 5.7.3. *Let $F$, $P$, and $F_\circ$ be as defined in Section 5.4, and $M$ as in Notation 5.3.5. Let $X_\alpha = \det^{-1}([-\alpha, \alpha]) = \{ x \in \mathrm{M}_n(\mathbb{R}) \mid |\det x| \leq \alpha \}$ and*

$$\mathcal{A}_{\alpha,\beta} = \left\{ f \in L^2(\mathrm{M}_n(\mathbb{R})) \;\middle|\; \mathrm{supp}(f) \subseteq X_\alpha \;\; \text{and} \;\; \|\widehat{f}\big|_{\overline{F}\cdot\ell}\|_{2q}^2 \leq \beta \left\| \widehat{f}\big|_{\overline{F}\cdot\ell} \right\|_2, \forall \ell \in P \right\},$$

*where $\widehat{f}\big|_{\overline{F}\cdot\ell}$ denotes the restriction of $\widehat{f}$ to $\overline{F}\cdot\ell$. Let $\phi(\epsilon)$ denote the upper bound from Theorem 5.7.1 on the measure of sets $E_{p,p'}$. Then for $f \in \mathcal{A}_{\alpha,\beta}$, upper and lower frame bounds in 5.3.6 can be improved as*

$$|R|\,\|f\|_2^2 \leq \sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, f, \rho[\gamma,k]^{-1} g \,\rangle_{L^2(\mathrm{M}_n(\mathbb{R}))}\right|^2 \leq \left(1 + M\phi(\epsilon)^{1-1/q}\alpha^{n(1-1/q)}\beta\right) |R|\,\|f\|_2^2.$$

*In particular, as $\phi(\epsilon) \to 0$ as $\epsilon \to 0$, the upper bound can be made arbitrarily close to $|R|$ for signals on $\mathcal{A}_{\alpha,\beta}$.*

*Proof.* We begin by using all calculations from the proof of Theorem 5.3.6, which proves the lower frame bound and shows that

$$\begin{aligned} \sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, f, \rho[\gamma,k]^{-1} g \,\rangle_{L^2(A)}\right|^2 &= |R| \sum_{k\in P} \int_A |\widehat{f}(\chi_b)|^2 |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \\ &= |R| \sum_{k\in P}\sum_{\ell\in P} \int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 |\widehat{g}(\chi_{bk^{-1}})|^2 \, db, \end{aligned}$$

where the last equality follows as $(F, P)$ forms a tiling system. We can rewrite this as

$$|R| \sum_{k\in P} \int_{\overline{F}\cdot k} |\widehat{f}(\chi_b)|^2 \, db + |R| \sum_{k\in P} \sum_{\substack{\ell\in P \\ \ell\neq k}} \int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 |\widehat{g}(\chi_{bk^{-1}})|^2 \, db.$$

The first term is $|R|\,\|\widehat{f}\|_{L^2(A)}^2$, so we proceed by bounding the integral in the second term.

CLAIM 5.7.4. *For all $k \neq \ell \in P$, we have*

$$\int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 \, |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \;\leq\; \max_{h\in\overline{F}\cdot\ell} |\det h|^{n/p} \, \lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{\ell,k})^{1/p} \, \|\widehat{f}\,\Big|_{\overline{F}\cdot\ell}\|^2_{L^{2q}(A)}.$$

*Proof of claim.* As $k \neq \ell$, $\widehat{g} \neq 0$ if $b \in E_{\ell,k}$. Additionally, $\widehat{g} \leq 1$, so we have

$$\int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 \, |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \;\leq\; \int_A \mathbb{1}_{E_{\ell,k}}(b) \, |\widehat{f}\,\Big|_{\overline{F}\cdot\ell}(\chi_b)|^2 \, db.$$

For any $q > 1$, we apply Hölder's inequality to obtain

$$\begin{aligned}\int_A \mathbb{1}_{E_{\ell,k}}(b) \, |\widehat{f}\,\Big|_{\overline{F}\cdot\ell}(\chi_b)|^2 \, db \;&\leq\; \|\mathbb{1}_{E_{\ell,k}}\|_{L^p(A)} \, \| \, |\widehat{f}\,\Big|_{\overline{F}\cdot\ell}|^2 \, \|_{L^q(A)} \\ &=\; \|\mathbb{1}_{E_{\ell,k}}\|_{L^p(A)} \, \| \, \widehat{f}\,\Big|_{\overline{F}\cdot\ell} \, \|^2_{L^{2q}(A)}.\end{aligned}$$

It only remains to bound $\|\mathbb{1}_{E_{\ell,k}}\|_{L^p(A)}$, which follows from

$$\begin{aligned}\|\mathbb{1}_{E_{\ell,k}}\|^p_{L^p(A)} &= \int_A \mathbb{1}_{E_{\ell,k}}(x) \, dx \\ &= \int_{E_{\ell,k}} |\det x|^n \, \frac{dx}{|\det x|^n} \\ &\leq \max_{h\in\overline{F}\cdot\ell} |\det h|^n \, \lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{\ell,k}).\end{aligned}$$

This establishes the claim. □

To complete the main proof, we apply Claim 5.7.4 and obtain

$$\begin{aligned}\sum_{\substack{k\in P\\ \ell\in P\\ \ell\neq k}} \int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \;&\leq\; \sum_{\substack{k\in P\\ \ell\in P\\ \ell\neq k}} \max_{h\in\overline{F}\cdot\ell} |\det h|^{n/p} \, \lambda_{\mathrm{GL}_n(\mathbb{R})}(E_{\ell,k})^{1/p} \, \|\widehat{f}\,\Big|_{\overline{F}\cdot\ell}\|^2_{L^{2q}(A)} \\ &\leq\; \alpha^{n/p} \, \phi(\epsilon)^{1/p} \sum_{k\in P} \sum_{\substack{\ell\in P\\ \ell\neq k}} \|\widehat{f}\,\Big|_{\overline{F}\cdot\ell}\|^2_{L^{2q}(A)}\end{aligned}$$

Applying this inequality with the fact that there are at most $(M-1)$ elements $\ell \in P$ such that $E_{\ell,k} \neq \emptyset$ for $k \neq \ell$, we conclude

$$|R| \sum_{k\in P} \sum_{\substack{\ell\in P\\ \ell\neq k}} \int_{\overline{F}\cdot\ell} |\widehat{f}(\chi_b)|^2 |\widehat{g}(\chi_{bk^{-1}})|^2 \, db \;\leq\; |R| \alpha^{n/p} \, \phi(\epsilon)^{1/p} \, (M-1)\beta \;\; \|f\|_2^2 \, .$$

□

Using reverse versions of Jensen's inequality, we hope to find natural descriptions for the spaces $\mathcal{A}_{\alpha,\beta}$ in the future.

## 5.8 Extensions to More General Function Spaces

In this section, we investigate if our previous results and methods could be used to form frames for Sobolev spaces.

**Classical Sobolev Spaces.** We now consider the classical Sobolev spaces, a class of weighted $L^2$ spaces, although we hope to extend results from this section to more general spaces in future work. We define the Sobolev space $H^s$ (for $s \in \mathbb{Z}^{\geq 0}$) as

$$H^s(\mathrm{M}_n(\mathbb{R})) := \left\{ f \in L^2(\mathrm{M}_n(\mathbb{R})) \;\middle|\; \mathcal{F}^{-1}\left[(1+|2\pi\xi|^2)^{s/2}\mathcal{F}f\right] \in L^2(\mathrm{M}_n(\mathbb{R})) \right\}.$$

Then $H^s(\mathrm{M}_n(\mathbb{R}))$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{H^s(\mathrm{M}_n(\mathbb{R}))} = \int_{\widehat{\mathrm{M}_n(\mathbb{R})}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}(1+(2\pi|\xi|)^2)^s \, d\xi,$$

and $H^s(\mathrm{M}_n(\mathbb{R}))$ is a Banach space with respect to the norm

$$\|f\|_{H^s(\mathrm{M}_n(\mathbb{R}))} = \left( \int_{\widehat{\mathrm{M}_n(\mathbb{R})}} |\widehat{f}(\xi)|^2(1+(2\pi|\xi|)^2)^s \, d\xi \right)^{1/2}.$$

For notational convenience, we will define the weight function as

$$\omega(\xi) = (1+(2\pi|\xi|)^2)^{s/2}.$$

While it is not obvious from the Fourier characterization of Sobolev spaces, an equivalent formulation of the space can be defined as a function $f$ is in $H^s(\mathrm{M}_n(\mathbb{R}))$ if and only if $f \in L^2(\mathrm{M}_n(\mathbb{R}))$ and all of its partial derivatives up to order $s$ are also in $L^2$. That is, the Sobolev norm captures a measure of how smooth a function is through the inclusion of derivative information.

**The Hilbert Frame in Sobolev Space.** We now explicitly give the calculation similar to that in Theorem 5.3.6, but replace the inner product with that for Sobolev spaces. Note that if we repeat the calculation as we did in 5.3.6 and use the unitary property of the representation to shift the action of the representation away from the wavelet to the function $f$, then we get

$$\sum_{k\in P}\sum_{\gamma\in J}\left|\langle\, f,\rho[\gamma,k]^{-1}g\,\rangle_{H^s(A)}\right|^2 = |R|\int_A |\widehat{f}(\chi_b)|^2\left(\sum_{k\in P}|\widehat{g}(\chi_{bk^{-1}})|^2\,\omega^4(\chi_{bk^{-1}})\right)db.$$

In this case, the weight function being disassociated with the function $f$ makes the Sobolev norm vanish from the computations, which is not useful for determining the frame requirements.

THEOREM 5.8.1. *Let $(F,P)$ be a tiling system for $\mathrm{GL}_n(\mathbb{R})$, with $R$ and $F_\circ$ as in Notation 5.3.5. Let $g\in H^s(\mathrm{M}_n(\mathbb{R}))$ be such that $\mathbb{1}_{\overline{F}}\le\widehat{g}\le\mathbb{1}_{F_\circ}$. Then the collection $\left\{\,\rho[\lambda,p]^{-1}g \;\middle|\; (\lambda,p)\in J\times P\right\}$ is a discrete frame in $H^s(\mathrm{M}_n(\mathbb{R}))$, where $J$ is defined in Notation 5.3.4, if and only if $s=0$.*

*Proof.* For an arbitrary $f\in H^s(\mathrm{M}_n(\mathbb{R}))$, we have

$$\begin{aligned}\sum_{k\in P}\sum_{\gamma\in J}\left|\langle\, f,\rho[\gamma,k]^{-1}g\,\rangle_{H^s(A)}\right|^2 &= \sum_{k\in P}\sum_{\gamma\in J}\left|\langle\,\widehat{f},\pi[-k^{-1}\gamma,k^{-1}]\widehat{g}\,\rangle_{H^s(A)}\right|^2\\ &= \sum_{k\in P}\sum_{\gamma\in J}\left|\int_A \widehat{f}(\chi_b)\,|\det k|^{-n/2}\chi_b(-k^{-1}\gamma)\overline{\widehat{g}(\chi_{bk^{-1}})}\omega^2(\chi_b)\,db\right|^2\\ &= \sum_{k\in P}\sum_{\gamma\in J}\left|\int_A \widehat{f}(\chi_b)\,|\det k|^{-n/2}\overline{\chi_{bk^{-1}}(\gamma)\widehat{g}(\chi_{bk^{-1}})}\omega^2(\chi_b)\,db\right|^2\\ &= \sum_{k\in P}\sum_{\gamma\in J}\left|\int_A \widehat{f}(\chi_{bk})\omega^2(\chi_{bk})\,|\det k|^{n/2}\overline{\chi_b(\gamma)\widehat{g}(\chi_b)}\,db\right|^2\end{aligned}$$

We proceed by defining the new function $h_k(b)=|\det k|^{n/2}\,\widehat{f}(\chi_{bk})\,\omega^2(\chi_{bk})\,\overline{\widehat{g}(\chi_b)}$ and noting once more that the set $\left\{ e_\gamma(b) = \chi_b(\gamma)\, \mathbb{1}_R(b)\, |R|^{-1/2} \;\middle|\; \gamma \in J \right\}$ forms an orthonormal basis for $L^2(R)$. Then applying Parseval's identity on $L^2(R)$ we see that

$$\begin{aligned}
\sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, f, \rho[\gamma,k]^{-1} g\,\rangle_{H^s(A)}\right|^2 &= |R| \sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, h_k, e_\gamma\,\rangle_{L^2(R)}\right|^2 \\
&= |R| \sum_{k\in P} \int_R |\det k|^n\, |\widehat{f}(\chi_{bk})|^2\, \omega^4(\chi_{bk})\, |\widehat{g}(\chi_b)|^2\, db \\
&= |R| \sum_{k\in P} \int_A |\widehat{f}(\chi_b)|^2\, \omega^4(\chi_b)\, |\widehat{g}(\chi_{bk^{-1}})|^2\, db \\
&= |R| \int_A |\widehat{f}(\chi_b)|^2\, \omega^4(\chi_b) \left( \sum_{k\in P} |\widehat{g}(\chi_{bk^{-1}})|^2 \right) db.
\end{aligned}$$

By the same argument as in Theorem 5.3.6, we obtain

$$|R|\, \|f\|^2_{H^{2s}(\mathrm{M}_n(\mathbb{R}))} \le \sum_{k\in P}\sum_{\gamma\in J} \left|\langle\, f, \rho[\gamma,k\,\rangle^{-1} g]_{H^s(\mathrm{M}_n(\mathbb{R}))}\right|^2 \le M|R|\, \|f\|^2_{H^{2s}(\mathrm{M}_n(\mathbb{R}))},$$

which clearly demonstrates that for any function $f \in H^s(\mathrm{M}_n(\mathbb{R}))$ which is not also in $H^{2s}(\mathrm{M}_n(\mathbb{R}))$, the lower bound is infinite so no upper frame bound is possible with respect to the Hilbert space structure for this construction. □

We also believe that the above proof holds for more general classes of weighted $L^2(\mathrm{M}_n(\mathbb{R}))$ spaces (such as Bessel potential spaces) of which the Sobolev spaces are a special case. However, several technical assumptions remain to be checked with regards to these spaces.

| $\kappa' - \kappa$ | $\lambda' - \lambda$ | $2^{2(\kappa'-\kappa)}\mu - \mu'$ | $\lambda_{\mathrm{M}_n(\mathbb{R})}$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $6\pi\log(2)$ |
| | | | |
| $0$ | $1$ | $0$ | $[6\pi]\log\left(\frac{1}{1-\epsilon}\right)$ |
| $0$ | $-1$ | $0$ | $[6\pi]\log\left(\frac{2+\epsilon}{2}\right)$ |
| | | | |
| $0$ | $0$ | $\{-1,1\}$ | $[6\pi\log(2)]\epsilon$ |
| $-1$ | $0$ | $\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$ | $[2\pi\log(2)]\,e - [\pi\log(2)]e^2$ |
| $1$ | $0$ | $\{-1,-2\}$ | $[8\pi\log(2)]\epsilon + [18\pi\log(2)]\epsilon^2 + [4\pi\log(2)]\epsilon^3$ |
| $1$ | $0$ | $\{0,-3\}$ | $[8\pi\log(2)]\epsilon + [10\pi\log(2)]e^2 + [2\pi\log(2)]\epsilon^3$ |
| | | | |
| $0$ | $1$ | $\{-1,1\}$ | $[6\pi]\epsilon\log\left(\frac{1}{1-\epsilon}\right)$ |
| $0$ | $-1$ | $\{-1,1\}$ | $[6\pi]\epsilon\log\left(\frac{2+\epsilon}{2}\right)$ |
| $-1$ | $1$ | $\{0,1/4,1/2,3/4\}$ | $[\pi]\epsilon\log\left(\frac{1}{1-\epsilon}\right) + \left[\frac{\pi}{2}\right]\epsilon^2\log\left(\frac{1}{1-\epsilon}\right)$ |
| $-1$ | $-1$ | $\{0,1/4,1/2,3/4\}$ | $[\pi]\epsilon\log\left(\frac{2+\epsilon}{2}\right) - \left[\frac{\pi}{2}\right]\epsilon^2\log\left(\frac{2+\epsilon}{2}\right)$ |
| $1$ | $1$ | $\{0,-3\}$ | $[8\pi]\epsilon\log\left(\frac{1}{1-\epsilon}\right) + [10\pi]\epsilon^2\log\left(\frac{1}{1-\epsilon}\right) + [2\pi]\epsilon^3\log\left(\frac{1}{1-\epsilon}\right)$ |
| $1$ | $1$ | $\{-1,-2\}$ | $[8\pi]\epsilon\log\left(\frac{1}{1-\epsilon}\right) + [18\pi]\epsilon^2\log\left(\frac{1}{1-\epsilon}\right) + [4\pi]\epsilon^3\log\left(\frac{1}{1-\epsilon}\right)$ |
| $1$ | $-1$ | $\{0,-3\}$ | $[8\pi]\epsilon\log\left(\frac{2+\epsilon}{2}\right) + [10\pi]\epsilon^2\log\left(\frac{2+\epsilon}{2}\right) + [2\pi]\epsilon^3\log\left(\frac{2+\epsilon}{2}\right)$ |
| $1$ | $-1$ | $\{-1,-2\}$ | $[8\pi]\epsilon\log\left(\frac{2+\epsilon}{2}\right) + [18\pi]\epsilon^2\log\left(\frac{2+\epsilon}{2}\right) + [4\pi]\epsilon^3\log\left(\frac{2+\epsilon}{2}\right)$ |
| | | | |
| $-1$ | $0$ | $\{-1/4,1\}$ | $[4\pi\log(2)]\epsilon^2 - [2\pi\log(2)]\epsilon^3$ |
| $1$ | $0$ | $\{-4,1\}$ | $[8\pi\log(2)]\epsilon^2 + [2\pi\log(2)]\epsilon^3$ |
| | | | |
| $-1$ | $1$ | $\{-1/4,1\}$ | $[4\pi]\epsilon^2\log\left(\frac{1}{1-\epsilon}\right) - [2\pi]\epsilon^3\log\left(\frac{1}{1-\epsilon}\right)$ |
| $1$ | $1$ | $\{-4,1\}$ | $[8\pi]\epsilon^2\log\left(\frac{1}{1-\epsilon}\right) + [2\pi]\epsilon^3\log\left(\frac{1}{1-\epsilon}\right)$ |
| $-1$ | $-1$ | $\{-1/4,1\}$ | $[4\pi]\epsilon^2\log\left(\frac{2+\epsilon}{2}\right) - [2\pi]\epsilon^3\log\left(\frac{2+\epsilon}{2}\right)$ |
| $1$ | $-1$ | $\{-4,1\}$ | $[8\pi]\epsilon^2\log\left(\frac{2+\epsilon}{2}\right) + [2\pi]\epsilon^3\log\left(\frac{2+\epsilon}{2}\right)$ |

**Table 5.2:** Measures of all intersecting sets for $n = 2$ and $\epsilon < \frac{1}{4}$.

## Chapter 6

## OUTLOOK AND OPEN PROBLEMS

In this chapter, we provide an overview of some open problems and planned future directions of the work already completed in Chapters 4 and 5. This provides some additional context for the work presented in this thesis, as well as some topics that may be of interest to other researchers. We include discussion, partial results, and conjectures, along with relevant references.

# 6.1 Open Problems on Graph Signal Processing

### Frames for General Cayley Graphs

In the case where a graph $\Gamma = \mathrm{Cay}(G, S)$ is a Cayley graph, but the generating set $S$ is not closed under conjugation, Theorem 4.4.3 does not apply. However, the subspaces $\mathcal{E}_{\pi,i}$ of $L^2(G)$ defined in the Peter-Weyl theorem (Theorem 2.1.6) are still invariant spaces of dimension $d_\pi$. Consequently, the basis in (4.20) block diagonalizes the adjacency (or Laplacian) matrix of *any* Cayley graph. This does not immediately yield a basis of eigenvectors, but it does imply that a basis of eigenvectors can always be found of the form

$$\phi_j^{(k)} = \sum_{j=1}^{d_\pi} c_j^{(k)} \pi_{i,j}, \qquad \text{for } k = 1, 2, \ldots, d_\pi, \tag{6.1}$$

for some coefficients $c_j^{(k)} \in \mathbb{C}$.

Let $\Gamma = \mathrm{Cay}(G;S)$ be a Cayley graph with an arbitrary generating set. Let $\Phi$ be the basis of normalized coefficient functions given in (4.20). Then (with the proper ordering of the vectors)

$$A_\Gamma = \Phi B \Phi^*, \tag{6.2}$$

where $B$ is block diagonal. Let us make a few observations which show we know a great deal about the block structure of the matrix $B$.

First notice that in our proof of Theorem 4.4.3, we did not use the condition that $S$ is closed under conjugation until the very last equality. Therefore, in the general case, it is still true that

$$A_\Gamma \pi_{i,j} = \sum_{s\in S} \sum_{k=1}^{d_\pi} \pi_{k,j}(s)\pi_{i,k},$$

which immediately demonstrates two facts: the entries in each block can be derived from the coefficient functions of a unique representation and the generating set, and that each block associated with the same representation has identical entries.

This already makes the problem for Cayley graphs substantially more tractable than the general case for the following reasons.

1. Let $\pi$ be a representation. We have already observed that each $d_\pi$ blocks associated to $\pi$ are identical. Consequently, the coefficients $c_j$ in (6.1) apply to each fixed $\pi$ and $i$ independent of $k$. That is,

   $$\phi_j^{(k)} = \sum_{j=1} c_j\, \pi_{i,j}, \qquad \text{for } k = 1, 2, \ldots, d_\pi,$$

   is an eigenvector for the same coefficients $\{c_j\}_{j=1}^{d_\pi}$. Then finding eigenvectors for one block involves diagonalizing a $d_\pi \times d_\pi$ matrix but yields $d_\pi^2$ eigenvectors for the adjacency or Laplacian matrix.

2. These yield eigenvectors for the graph. If $A_\Gamma$ decomposes as in (6.2), and $v$ is an eigenvector of $B$, then $\Phi v$ is an eigenvector of $A$ associated to the same eigenvalue. So finding eigenvectors on the blocks leads to an eigen decomposition of the whole space.

3. The maximum size of any block would be $\lfloor\sqrt{N-1}\rfloor \times \lfloor\sqrt{N-1}\rfloor$, as the trivial representation is always present (it is associated to the eigenvector of all ones).

We conjecture that an eigenbasis of the form in (6.1) could yield results similar to Theorem 4.4.6; that specific family of translations could be simplified in terms of the group structure for this basis. Additionally, we suspect that the condition of Theorem 4.4.6 can be relaxed from needing the Fourier transform of the localizing function to be constant on representations to possibly being constant on a row or column of the representation, but our current proof technique does not work in this case.

## Joint Localization

One of the primary advantages of Gabor-type frames for functions defined on Euclidean spaces $\mathbb{R}^n$ is that the time windows can be chosen to have good localization properties in both time and frequency domains, and this property is preserved by their shifts and modulations. However, understanding the support of window functions in both domains based on the graph Fourier transform proves to be a much more complicated issue due to the irregular domain represented by the graph. This is partially due to the fact that the common choice of modulation operator fails to correspond to shifts in the frequency domain as it does in the classical case shown in Proposition 2.2.6.

The above observation leads to an alternative definition for modulation by Thanh, Linh-Trung, Dung, and Abed-Meraim in [111], which is defined as cyclic shifts in the spectral domain. It would be interesting to see if a general frame construction as in Theorem 4.3.3 could be proved with alternative definitions of modulation like this one, which would make understanding the joint localization properties much easier. Additionally, one could attempt to classify which time window functions would provide good joint localization properties with regards to any of the current constructions.

In Chapter 4, we looked at different notions of graph shift operators. In the vertex domain, we had approximations to permutations which would allow some vertices to vanish, as in [62, 91], which makes the localization in the vertex domain easy to understand. Similarly, shifting defined through polynomials of the adjacency (as in [94]) is clearly understood in the vertex domain to have a diffusive effect. However, it is much less clear when using the convolutive shift operators defined in the spectral domain, such as [53, 103]. The authors in [103] provide bounds on localization by first finding bounds when the localizing function is defined in the graph spectral domain as a polynomial of the eigenvalues. Then, by using polynomial approximations for sufficiently smooth functions, they provide bounds on how well localized a given function will be, but these techniques require specific choices for the time window $\mathfrak{g} : V \to \mathbb{C}$.

In all cases, there are several desirable properties for the graph shift operators, which all hold in the classical framework:

(P1) The shift operators are isometric (energy preserving).

(P2) The shift operators respect the graph structure (neighborhood preserving).

(P3) The shift operators act sharply 1-transitively on the vertex set (there is a unique shift sending vertex $i$ to vertex $j$ for every pair $i$ and $j$).

(P4) The shift operators form a group.

In Section 4.4 we saw that, in the special case of Cayley graphs, there is a natural way of defining graph shifts which achieve all of these goals. Denote the group of graph automorphisms by $\mathrm{Aut}(\Gamma)$. Then by noting that the left regular representations satisfy $\{ L(g) \mid g \in G \} \subseteq \mathrm{Aut}(\Gamma)$, we leveraged the group structure to find graph shifts which were graph automorphisms and satisfied all of the above properties. It is not hard to compute the vector $v$ in Theorem 4.3.3 in the case that shifts are sharply 1-transitive permutations. In this case, $\{M_\ell A_k \mathfrak{g}\}$ forms a tight frame with frame bound $|V(\Gamma)| \cdot \|\mathfrak{g}\|^2$ for any choice of function $\mathfrak{g}$.

Observe that properties (P3) and (P4) imply that no shift operator fixes any point except for the identity operator. If we replace condition (P4) with

(P4') If $\varrho \in \mathrm{Aut}(\Gamma)$ fixes any point, then $\varrho$ is the identity automorphism, then we can still classify all graphs which satisfy (P1)-(P3) and the weaker condition (P4)' due to the following theorem of Gauyacq from 1997.

THEOREM 6.1.1 ([52], Theorem 1). *A connected graph $\Gamma$ is quasi-Cayley if and only if the automorphism group of $\Gamma$ contains a sharply 1-transitive subset acting on $V(\Gamma)$.*

The important property for a quasi-group $Q$ is that, for any two elements $a, b$ in $Q$, $ax = b$ and $ya = b$ each have unique solutions $x$ and $y$. In other words, the left- and right-shift operators are still permutations which act transitively on the group. Quasi-Cayley graphs are defined similarly to Cayley graphs in Section 4.4, however we replace the group $G$ with a quasi-group which contains a left-identity element (there exists $x \in Q$ such that $xq = q$ for all $q$ in $Q$), and the generating set $S$ must be left-associative $\big($for all $a$ and $b$ in $Q$, $S(ab) = (Sa)b\big)$. Then the vertex set is $Q$ and the edge-set is given by $\left\{ (q, sq) \;\middle|\; q \in Q, s \in S \right\}$, exactly as it was in the case of Cayley graphs.

This leads to two interesting problems. First

PROBLEM 6.1.2. *Does there exist a natural choice of basis vectors for quasi-Cayley graphs which would lead to properties like those in Corollary 4.4.7? In particular, can we choose a basis such that previously defined translations behave like left-shift operators from the quasi-group?*

Second, can we find a translation operator which behaves like left-shift operators in the basis of coefficient functions so that it always reduces to left-shift translations for Cayley graphs?

PROBLEM 6.1.3. *Does there exist a way to define a general translation operator on arbitrary graphs which generalizes the left-shift automorphisms for Cayley graphs when using the basis from* (6.1)*?*

We end the discussion of joint localization with the smallest possible example of a graph which is quasi-Cayley but not Cayley. This is the famous Petersen graph,

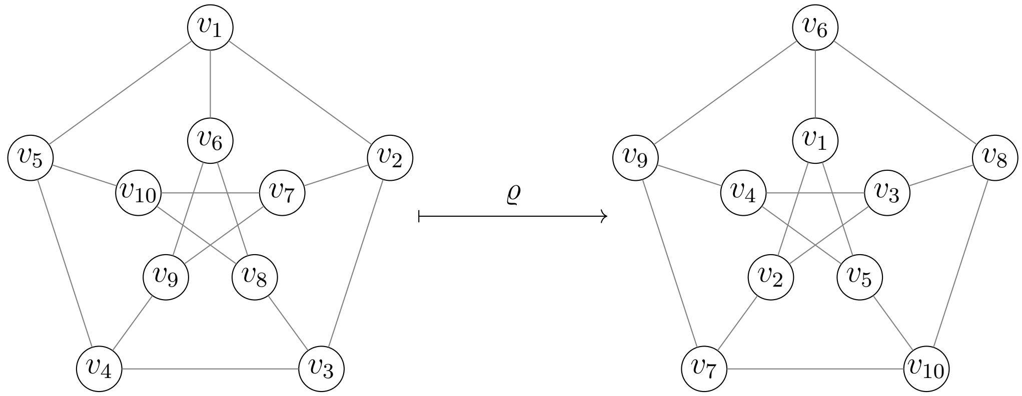



**Figure 6.1:** The Petersen graph with an automorphism which inverts inner and outer cycles.

and we describe the natural choice of translations based on the graph symmetries for it as well.

Example 6.1.4. Let $\Gamma$ be the Petersen graph illustrated in Figure 6.1. It is well known that this is not Cayley. However, it can be realized as a quasi-Cayley graph of the quasi-group $\mathbb{Z}_2 \times \mathbb{Z}_5$ with group operation

$$(x, y) \oplus (x', y') = (x + x' \mod 2, y + 2^x y' \mod 5)$$

for the generating set $S = \{(1,0),(0,1),(0,4)\}$ (cf. [52, 88] for construction details).

The automorphism group of this graph is also isomorphic to $S_5$, the symmetric group of permutations on 5 objects. In this case, taking the automorphism $\varrho$ illustrated in 6.1 and the automorphism $\varphi$ which shifts labels 72°, one can take the shift operators as:

$$\left\{ \varphi^k, \varphi^k \varrho \;\middle|\; k = 1, 2, 3, 4, 5 \right\}.$$

It is easy to check that $\varrho$ has order 4, so it cannot belong to a group of order 10. So for the Petersen graph, this collection of automorphisms satisfies conditions (P1)-(P3) and (P4)', but it does not form a group.

It would be interesting to find the best approximations to this structure for arbitrary graphs. Currently, it seems the best strategy is to exploit features for specific families of graphs which might arise in applications.

## 6.2 Open Problems for the Discretization Problem

### Constructions for Other Matrix Subgroups

In [29], Currey, Führ, and Oussa classify all the $3 \times 3$ matrix groups in terms of whether or not they admit a continuous wavelet transform of pure translates. A natural question, then, would be for those which do not admit a frame of pure translates, how many admit a tiling system as in Definition 5.3.1?

This, of course, leads to many other interesting questions:

1. For those that do admit a tiling system, how can it be constructed?

2. Once constructed, does the discretization of the space always lead to a discrete wavelet frame?

3. For any such construction, is there always a way to generalize it to higher-dimensional versions of the group?

As mentioned in Section 3.1, the Gabor and wavelet frames correspond to the action of an irreducible representation from the Heisenberg and affine groups, respectively. These representations appear most frequently in applications, but considering continuous wavelet transforms arising from the irreducible representations of other groups classified in the above work might lead to a better understanding of discrete frame constructions.

**Extending Results of Chapter 5 to Banach Frames** Beginning in the late 1980's with the series of papers [37, 38, 39], Feichtinger and Gröchenig began a systematic investigation of series expansions of functions for Banach spaces. Here we provide a brief overview of this work, as it provides a promising direction to generalize the discrete frame constructed in Chapter 5 to weighted $L^2(\mathrm{M}_n(\mathbb{R}))$ spaces $\big($recall that we have already shown this does not generalize as a Hilbert space frame (see Theorem 5.8.1)$\big)$.

For the remainder of this chapter, let $X$ be a Banach space with dual space $X^*$ (the space of continuous linear functions from $X$ to $\mathbb{C}$). The definition of an *atomic decomposition* of a Banach space relies on a *Banach sequence space* consisting of sequences of complex numbers $\{c_k\}_{k=1}^{\infty}$ which we will denote by $X_d$.

DEFINITION 6.2.1 (Atomic Decomposition). Let $X$ be a Banach space and $X_d$ be a Banach sequence space indexed by $\mathbb{N}$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $X$ and $\{g_k\}_{k=1}^{\infty}$ be a sequence in $X^*$. Then the pair $\left(\{g_k\}_{k=1}^{\infty}, \{f_k\}_{k=1}^{\infty}\right)$ is an atomic decomposition of $X$ with respect to $X_d$ if

1. $\{g_k(f)\}_{k=1}^{\infty} \in X_d$ for all $f \in X$;
2. there exist positive constants $A$ and $B$ such that

$$A\,\|f\|_X \;\leq\; \|\{g_k(f)\}_{k=1}^{\infty}\|_{X_d} \;\leq\; B\,\|f\|_X\,, \qquad \forall f \in X;$$

3. $f = \sum_{k=1}^{\infty} g_k(f) f_k$, for all $f \in X$.

REMARK 6.2.2. Note that in Hilbert spaces, the second condition ensures a stable reconstruction method (condition 3), however, in Banach spaces they are not necessarily related. For a complete discussion of atomic decompositions versus frames, we direct the interested reader to [63].

DEFINITION 6.2.3 (Banach Frame). Let $X$ be a Banach space and $X_d$ a Banach sequence space indexed by $\mathbb{N}$. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence in $X^*$ and $S : X_d \to X$ be a bounded operator. Then $\left(\{g_k\}_{k=1}^{\infty}, S\right)$ is a Banach frame for $X$ with respect to $X_d$ if

1. $\{g_k(f)\}_{k=1}^{\infty} \in X_d$ for all $f \in X$;
2. there exist positive constants $A$ and $B$ such that

$$A\,\|f\|_X \;\leq\; \|\{g_k(f)\}_{k=1}^{\infty}\|_{X_d} \;\leq\; B\,\|f\|_X\,, \qquad \forall f \in X;$$

3. $S\{g_k(f)\} = f$, for all $f \in X$.

REMARK 6.2.4. The key difference between atomic decompositions and Banach frames is that atomic decompositions ask for a series expansion for each element of the Banach space, whereas Banach frames have a bounded synthesis operator $S$ which allows for reconstruction from the coefficients of the original signal, not necessarily as an infinite series. This is a subtle, but important difference, because one can have Banach frames for Hilbert spaces which are not simultaneously Hilbert frames, since the sequence space $X_d$ is not necessarily $\ell^2$ (consider Example 3.2.11, which provides a Banach frame with sequence space $\ell^\infty$ which is not an atomic decomposition).

To verify that the construction of Chapter 5 satisfies the conditions of a Banach frame, we would replace the inner product with the duality pairing between $X$ and $X^*$, and verify that this leads to a bounded, invertible operator $S$ as required in Definition 6.2.3. For a parallel construction for Gabor systems, see [10, Chapter 5].

Other Banach frame constructions based on Riesz sequences have been given by Borup and Nielsen in [13, 14]. It is worth noting that the tight frame constructions here rely on infinitely many generating wavelet functions, and tight frames are lost when restricting to finitely generated systems. Fornasier studies wavelet and Gabor frames for generalized $\alpha$-modulation spaces using tools from self-localized frame theory in [42]. Borup, Gribonval, and Nielsen also obtained atomic decompositions for $L^p(\mathbb{R}^d)$ and Sobolev spaces by using completely different methods in [12].

# Bibliography

[1] Hossein Abbaspour and Martin Moskowitz. *Basic Lie Theory.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[2] Syed Twareque Ali, Jean-Pierre Antoine, and Jean-Pierre Gazeau. Continuous frames in Hilbert space. *Ann. Physics*, 222(1):1–37, 1993.

[3] Syed Twareque Ali, Jean-Pierre Antoine, and Jean-Pierre Gazeau. *Coherent States, Wavelets and Their Generalizations.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000.

[4] Syed Twareque Ali and Kengatharam Thirulogasanthar. The quaternionic affine group and related continuous wavelet transforms on complex and quaternionic Hilbert spaces. *J. Math. Phys.*, 55(6):063501, 16, 2014.

[5] Paolo Aniello, Gianni Cassinelli, Ernesto De Vito, and Alberto Levrero. On discrete frames associated with semidirect products. *J. Fourier Anal. Appl.*, 7(2):199–206, 2001.

[6] László Babai. Spectra of Cayley graphs. *J. Combin. Theory Ser. B*, 27(2):180–189, 1979.

[7] Matthew Begué and Kasso A. Okoudjou. Invertibility of graph translation and support of Laplacian Fiedler vectors. In *Frames and harmonic analysis*, volume 706 of *Contemp. Math.*, pages 153–174. Amer. Math. Soc., Providence, RI, 2018.

[8] Hamid Behjat, Ulrike Richter, Dimitri Van De Ville, and Leif Sörnmo. Signal-adapted tight frames on graphs. *IEEE Trans. Signal Process.*, 64(22):6017–6029, 2016.

[9] Hamid Behjat and Dimitri Van De Ville. Spectral design of signal-adapted tight frames on graphs. In *Vertex-frequency analysis of graph signals*, Signals Commun. Technol., pages 177–206. Springer, Cham, 2019.

[10] John J. Benedetto and David F. Walnut. Gabor frames for $L^2$ and related spaces. In *Wavelets: mathematics and applications*, Stud. Adv. Math., pages 97–162. CRC, Boca Raton, FL, 1994.

[11] David Bernier and Keith F. Taylor. Wavelets from square-integrable representations. *SIAM J. Math. Anal.*, 27(2):594–608, 1996.

[12] Lasse Borup, Rémi Gribonval, and Morton Nielsen. Tight wavelet frames in Lebesgue and Sobolev spaces. *J. Funct. Spaces Appl.*, 2(3):227–252, 2004.

[13] Lasse Borup and Morten Nielsen. Banach frames for multivariate $\alpha$-modulation spaces. *J. Math. Anal. Appl.*, 321(2):880–895, 2006.

[14] Lasse Borup and Morten Nielsen. Frame decomposition of decomposition spaces. *J. Fourier Anal. Appl.*, 13(1):39–70, 2007.

[15] Andries E. Brouwer and Willem H. Haemers. *Spectra of Graphs.* Universitext. Springer, New York, 2012.

[16] Alberto P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.*, 24:113–190, 1964.

[17] Emmanuel J. Candès and David L. Donoho. Curvelets and curvilinear integrals. *J. Approx. Theory*, 113(1):59–90, 2001.

[18] Emmanuel J. Candès and David L. Donoho. Continuous curvelet transform. I. Resolution of the wavefront set. *Appl. Comput. Harmon. Anal.*, 19(2):162–197, 2005.

[19] Emmanuel J. Candès and David L. Donoho. Continuous curvelet transform. II. Discretization and frames. *Appl. Comput. Harmon. Anal.*, 19(2):198–222, 2005.

[20] Siheng Chen, Rohan Varma, Aliaksei Sandryhaila, and Jelena Kovačević. Discrete signal processing on graphs: sampling theory. *IEEE Trans. Signal Process.*, 63(24):6510–6523, 2015.

[21] Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, [Cham], second edition, 2016.

[22] Ole Christensen, Baiqiao Deng, and Christopher Heil. Density of Gabor frames. *Appl. Comput. Harmon. Anal.*, 7(3):292–304, 1999.

[23] Charles K. Chui, Frank D. Filbir, and Hrushikesh N. Mhaskar. Representation of functions on big data: graphs and trees. *Appl. Comput. Harmon. Anal.*, 38(3):489–509, 2015.

[24] Samuel Cogar and Kris Hollingsworth. Discrete frames for Sobolev spaces arising from tiling systems on $\mathrm{GL}_n(\mathbb{R})$. In preparation.

[25] Ronald R. Coifman and Mauro Maggioni. Diffusion wavelets. *Appl. Comput. Harmon. Anal.*, 21(1):53–94, 2006.

[26] Elena Cordero, Filippo De Mari, Krzysztof Nowak, and Anita Tabacco. Dimensional upper bounds for admissible subgroups for the metaplectic representation. *Math. Nachr.*, 283(7):982–993, 2010.

[27] Elena Cordero and Anita Tabacco. Triangular subgroups of $Sp(d,\mathbb{R})$ and reproducing formulae. *J. Funct. Anal.*, 264(9):2034–2058, 2013.

[28] Mark Crovella and Eric Kolaczyk. Graph wavelets for spatial traffic analysis. In *INFOCOM 2003 Twenty-Second Annual Joint Conference of the IEEE Computer and Communications*, volume 3, pages 1848 – 1857 vol.3, 01 2003.

[29] Bradley Currey, Hartmut Führ, and Vignon Oussa. A classification of continuous wavelet transforms in dimension three. *Appl. Comput. Harmon. Anal.*, 46(3):500–543, 2019.

[30] Bradley Currey, Hartmut Führ, and Keith Taylor. Integrable wavelet transforms with abelian dilation groups. *J. Lie Theory*, 26(2):567–596, 2016.

[31] Ingrid Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.

[32] Ingrid Daubechies, Alexander Grossmann, and Yves Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, 1986.

[33] Bin Dong. Sparse representation on graphs by tight wavelet frames and applications. *Appl. Comput. Harmon. Anal.*, 42(3):452–479, 2017.

[34] Richard J. Duffin and Albert C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[35] Glenn Easley, Demetrio Labate, and Wang-Q Lim. Sparse directional image representations using the discrete shearlet transform. *Appl. Comput. Harmon. Anal.*, 25(1):25–46, 2008.

[36] Jacques Faraut. *Analysis on Lie Groups*, volume 110 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008. An introduction.

[37] Hans G. Feichtinger and Karlheinz Gröchenig. A unified approach to atomic decompositions via integrable group representations. In *Function spaces and*

*applications (Lund, 1986)*, volume 1302 of *Lecture Notes in Math.*, pages 52–73. Springer, Berlin, 1988.

[38] Hans G. Feichtinger and Karlheinz Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. I. *J. Funct. Anal.*, 86(2):307–340, 1989.

[39] Hans G. Feichtinger and Karlheinz Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. II. *Monatsh. Math.*, 108(2-3):129–148, 1989.

[40] Jonathan Fell, Hartmut Führ, and Felix Voigtlaender. Resolution of the wavefront set using general continuous wavelet transforms. *J. Fourier Anal. Appl.*, 22(5):997–1058, 2016.

[41] Gerald B. Folland. *A Course in Abstract Harmonic Analysis.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.

[42] Massimo Fornasier. Banach frames for $\alpha$-modulation spaces. *Appl. Comput. Harmon. Anal.*, 22(2):157–175, 2007.

[43] Massimo Fornasier and Holger Rauhut. Continuous frames, function spaces, and the discretization problem. *J. Fourier Anal. Appl.*, 11(3):245–287, 2005.

[44] Daniel Freeman and Darrin Speegle. The discretization problem for continuous frames. *Adv. Math.*, 345:784–813, 2019.

[45] Hartmut Führ. *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, volume 1863 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2005.

[46] Hartmut Führ. Generalized Calderón conditions and regular orbit spaces. *Colloq. Math.*, 120(1):103–126, 2010.

[47] Hartmut Führ. Coorbit spaces and wavelet coefficient decay over general dilation groups. *Trans. Amer. Math. Soc.*, 367(10):7373–7401, 2015.

[48] Hartmut Führ and Karlheinz Gröchenig. Sampling theorems on locally compact groups from oscillation estimates. *Math. Z.*, 255(1):177–194, 2007.

[49] Hartmut Führ and Matthias Mayer. Continuous wavelet transforms from semidirect products: cyclic representations and Plancherel measure. *J. Fourier Anal. Appl.*, 8(4):375–397, 2002.

[50] Hartmut Führ and Vignon Oussa. Groups with frames of translates, 02 2018.

[51] Dennis Gabor. Theory of communication. *Journal of the Institution of Electrical Engineers - Part III: Radio and Communication Engineering*, 93:429–457, November 1946.

[52] Ginette Gauyacq. On quasi-Cayley graphs. *Discrete Appl. Math.*, 77(1):43–58, 1997.

[53] Adnan Gavili and Xiao-Ping Zhang. On the shift operator, graph frequency, and optimal filtering in graph signal processing. *IEEE Trans. Signal Process.*, 65(23):6303–6318, 2017.

[54] Matan Gavish, Boaz Nadler, and Ronald Coifman. Multiscale wavelets on trees, graphs and high dimensional data: Theory and applications to semi supervised learning. In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, pages 367–374, 08 2010.

[55] Mahya Ghandehari, Dominique Guillot, and Kris Hollingsworth. A noncommutative viewpoint on graph signal processing. In *Proc. Int. Conf. Samp. Theory and Appl.*, Bordeaux, July 2019.

[56] Mahya Ghandehari, Dominique Guillot, and Kris Hollingsworth. Gabor-type frames for signal processing on graphs. In preparation.

[57] Mahya Ghandehari and Kris Hollingsworth. Discrete frames for $L^2(\mathbb{R}^{n^2})$ arising from tiling systems on $\mathrm{GL}_n(\mathbb{R})$. Submitted (arXiv preprint arXiv:2003.13113).

[58] Mahya Ghandehari, Aizhan Syzdykova, and Keith F. Taylor. A four dimensional continuous wavelet transform. In *Commutative and noncommutative harmonic analysis and applications*, volume 603 of *Contemp. Math.*, pages 123–136. Amer. Math. Soc., Providence, RI, 2013.

[59] Benjamin Girault, Paulo Gonçalves, and Éric Fleury. Translation on graphs: An isometric shift operator. *IEEE Signal Processing Letters*, 22(12):2416–2420, Dec 2015.

[60] Franziska Göbel, Gilles Blanchard, and Ulrike von Luxburg. Construction of tight frames on graphs and application to denoising. In *Handbook of big data analytics*, Springer Handb. Comput. Stat., pages 503–522. Springer, Cham, 2018.

[61] Francesco Grassi, Andreas Loukas, Nathanaël Perraudin, and Benjamin Ricaud. A time-vertex signal processing framework: scalable processing and meaningful representations for time-series on graphs. *IEEE Trans. Signal Process.*, 66(3):817–829, 2018.

[62] Nicolas Grelier, Bastien Pasdeloup, Jean-Charles Vialatte, and Vincent Gripon. Neighborhood-preserving translations on graphs. In *2016 IEEE Global Conference on Signal and Information Processing (GlobalSIP)*, pages 410–414, Dec 2016.

[63] Karlheinz Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(1):1–42, 1991.

[64] Alexander Grossmann and Jean Morlet. Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.*, 15(4):723–736, 1984.

[65] Kanghui Guo, Gitta Kutyniok, and Demetrio Labate. Sparse multidimensional representations using anisotropic dilation and shear operators. In *Wavelets and*

*splines: Athens 2005*, Mod. Methods Math., pages 189–201. Nashboro Press, Brentwood, TN, 2006.

[66] Kanghui Guo and Demetrio Labate. Optimally sparse multidimensional representation using shearlets. *SIAM J. Math. Anal.*, 39(1):298–318, 2007.

[67] David K. Hammond, Pierre Vandergheynst, and Rémi Gribonval. Wavelets on graphs via spectral graph theory. *Appl. Comput. Harmon. Anal.*, 30(2):129–150, 2011.

[68] Christopher E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *SIAM Rev.*, 31(4):628–666, 1989.

[69] Peter Heinlein. Discretizing continuous wavelet transforms using integrated wavelets. *Appl. Comput. Harmon. Anal.*, 14(3):238–256, 2003.

[70] Peter Heinlein, Johann Drexl, and Wilfried Schneider. Integrated wavelets for enhancement of microcalcifications in digital mammography. *IEEE Transactions on Medical Imaging*, 22(3):402–413, March 2003.

[71] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, second edition, 1979. Structure of topological groups, integration theory, group representations.

[72] Kris Hollingsworth. Improved frame bounds for a discrete frame construction on $L^2(\mathbb{R}^{n^2})$. In preparation.

[73] Saidhakim D. Ikramov. Matrix pencils: Theory, applications, and numerical methods. *Journal of Soviet Mathematics*, 64(2):783–853, 1993.

[74] Maarten Jansen, Guy P. Nason, and Bernard W. Silverman. Multiscale methods for data on graphs and irregular multidimensional situations. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 71(1):97–125, 2009.

[75] Iva Jestrović, James L. Coyle, and Ervin Sejdić. A fast algorithm for vertex-frequency representations of signals on graphs. *Signal Processing*, 131:483 – 491, 2017.

[76] Jay Jorgenson and Serge Lang. *Spherical Iinversion on* $SL_n(\mathbf{R})$. Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.

[77] Gerald Kaiser. *A Friendly Guide to Wavelets*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011. Reprint of the 1994 edition.

[78] Eberhard Kaniuth and Keith F. Taylor. *Induced Representations of Locally Compact Groups*, volume 197 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013.

[79] Gitta Kutyniok. *Affine Density in Wavelet Analysis*, volume 1914 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[80] Gitta Kutyniok and Demetrio Labate. Resolution of the wavefront set using continuous shearlets. *Trans. Amer. Math. Soc.*, 361(5):2719–2754, 2009.

[81] Richard S. Laugesen, Nik Weaver, Guido L. Weiss, and Edward N. Wilson. A characterization of the higher dimensional groups associated with continuous wavelets. *J. Geom. Anal.*, 12(1):89–102, 2002.

[82] Ann B. Lee, Boaz Nadler, and Larry Wasserman. Treelets—an adaptive multi-scale basis for sparse unordered data. *Ann. Appl. Stat.*, 2(2):435–471, 2008.

[83] Nora Leonardi and Dimitri Van De Ville. Tight wavelet frames on multislice graphs. *IEEE Trans. Signal Process.*, 61(13):3357–3367, 2013.

[84] Ren-Cang Li. Rayleigh quotient based optimization methods for eigenvalue problems. In *Matrix Functions and Matrix Equations*, pages 76–108. World Scientific, 2015.

[85] Joshua MacArthur and Keith F. Taylor. Wavelets with crystal symmetry shifts. *J. Fourier Anal. Appl.*, 17(6):1109–1118, 2011.

[86] Mauro Maggioni and Hrushikesh N. Mhaskar. Diffusion polynomial frames on metric measure spaces. *Appl. Comput. Harmon. Anal.*, 24(3):329–353, 2008.

[87] Fionn Murtagh. The Haar wavelet transform of a dendrogram. *J. Classification*, 24(1):3–32, 2007.

[88] Eric Mwambene. Cayley graphs on left quasi-groups and groupoids representing $k$-generalised Petersen graphs. *Discrete Math.*, 309(8):2544–2547, 2009.

[89] Kampanat Namngam and Eckart Schulz. Extensions of the Heisenberg group by one-parameter groups of dilations which are subgroups of the affine and the symplectic groups. *Math. Nachr.*, 288(2-3):309–326, 2015.

[90] Antonio Ortega, Pascal Frossard, Jelena Kovačević, José M. F. Moura, and Pierre Vandergheynst. Graph signal processing: Overview, challenges, and applications. *Proceedings of the IEEE*, 106(5):808–828, May 2018.

[91] Bastien Pasdeloup, Vincent Gripon, Jean-Charles Vialatte, Nicolas Grelier, and Dominique Pastor. A neighborhood-preserving translation operator on graphs, 2017.

[92] Nathanael Perraudin, Benjamin Ricaud, David I. Shuman, and Pierre Vandergheynst. Global and local uncertainty principles for signals on graphs. *APSIPA Transactions on Signal and Information Processing*, 7:e3, 2018.

[93] Walter Rudin. *Fourier Analysis on Groups*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.

[94] Aliaksei Sandryhaila and José M. F. Moura. Discrete signal processing on graphs. *IEEE Trans. Signal Process.*, 61(7):1644–1656, 2013.

[95] Aliaksei Sandryhaila and José M. F. Moura. Big data analysis with signal processing on graphs: Representation and processing of massive data sets with irregular structure. *IEEE Signal Processing Magazine*, 31(5):80–90, Sept 2014.

[96] Aliaksei Sandryhaila and José M. F. Moura. Discrete signal processing on graphs: frequency analysis. *IEEE Trans. Signal Process.*, 62(12):3042–3054, 2014.

[97] Erwin Schrödinger. Der stetige ubergang von der mikro- zur makromechanik. *Naturwissenschaften*, 14(4):664666, 1926.

[98] Eckart Schulz and Keith F. Taylor. Extensions of the Heisenberg group and wavelet analysis in the plane. In *Spline functions and the theory of wavelets (Montreal, PQ, 1996)*, volume 18 of *CRM Proc. Lecture Notes*, pages 217–225. Amer. Math. Soc., Providence, RI, 1999.

[99] Jean-Pierre Serre. *Linear Representations of Finite Groups.* Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[100] Rasoul Shafipour, Ali Khodabakhsh, and Gonzalo Mateos. A windowed digraph fourier transform. In *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 7525–7529, May 2019.

[101] David I. Shuman, Sunil K. Narang, Pascal Frossard, Antonio Ortega, and Pierre Vandergheynst. The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains. *IEEE Signal Processing Magazine*, 30(3):83–98, May 2013.

[102] David I. Shuman, Benjamin Ricaud, and Pierre Vandergheynst. A windowed graph fourier transform. In *2012 IEEE Statistical Signal Processing Workshop (SSP)*, pages 133–136, Aug 2012.

[103] David I. Shuman, Benjamin Ricaud, and Pierre Vandergheynst. Vertex-frequency analysis on graphs. *Appl. Comput. Harmon. Anal.*, 40(2):260–291, 2016.

[104] David I. Shuman, Christoph Wiesmeyr, Nicki Holighaus, and Pierre Vandergheynst. Spectrum-adapted tight graph wavelet and vertex-frequency frames. *IEEE Trans. Signal Process.*, 63(16):4223–4235, 2015.

[105] Ljubiša Stanković, Danilo P. Mandic, Miloš Daković, Bruno Scalzo, Miloš Brajović, Ervin Sejdić, and Anthony G. Constantinides. Vertex-frequency graph signal processing: A review, 2019.

[106] Ljubiša Stanković and Ervin Sejdić, editors. *Vertex-Frequency Analysis of Graph Signals*. Signals and Communication Technology. Springer, Cham, 2019.

[107] Jean-Luc Starck, Emmanuel J. Candès, and David L. Donoho. The curvelet transform for image denoising. *IEEE Trans. Image Process.*, 11(6):670–684, 2002.

[108] Jean-Luc Starck, Fionn Murtagh, Emmanuel J. Candès, and David L. Donoho. Gray and color image contrast enhancement by the curvelet transform. *IEEE Trans. Image Process.*, 12(6):706–717, 2003.

[109] Mariano Tepper and Guillermo Sapiro. A short-graph fourier transform via personalized PageRank vectors. In *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4806–4810, March 2016.

[110] Audrey Terras. *Fourier Analysis on Finite Groups and Applications*, volume 43 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[111] Le T. Thanh, Nguyen Linh-Trung, Nguyen V. Dung, and Karim Abed-Meraim. A new windowed graph Fourier transform. In *2017 4th NAFOSTED Conference on Information and Computer Science*, pages 150–155, Nov 2017.

[112] Dorina Thanou, David I. Shuman, and Pascal Frossard. Learning parametric dictionaries for signals on graphs. *IEEE Trans. Signal Process.*, 62(15):3849–3862, 2014.

[113] Mikhail Tsitsvero, Sergio Barbarossa, and Paolo Di Lorenzo. Signals on graphs: uncertainty principle and sampling. *IEEE Trans. Signal Process.*, 64(18):4845–4860, 2016.

[114] Robert M. Young. *An Introduction to Nonharmonic Fourier Series*, volume 93 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers],New York-London, 1980.

[115] Xian-Wei Zheng, Yuan Y. Tang, Jian-Tao Zhou, Hao-Liang Yuan, Yu-Long Wang, Li-Na Yang, and Jian-Jia Pan. Multi-windowed graph fourier frames. In *2016 International Conference on Machine Learning and Cybernetics (ICMLC)*, volume 2, pages 1042–1048, July 2016.